A Scalings Approach to $\mathcal{H}_2$-Gain-Scheduling Synthesis without Elimination ⋆

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Abstract: We present a direct synthesis approach to $\mathcal{H}_2$-gain-scheduling for time-varying parametric scheduling blocks with D- and positive real scalings based on a convexifying transformation for the controller parameters. In particular, finiteness of the $\mathcal{H}_2$-norm for the closed-loop system is achieved by solving a specific design problem with structured linear fractional representations of the plant and the controller. To enlarge the field of applications, we extend the framework to networks consisting of gain-scheduling systems with a delayed coupling.

Keywords: $\mathcal{H}_\infty/\mathcal{H}_2$ control, optimal control, network controlled-systems

1. INTRODUCTION

Gain-scheduled controller design plays an important role in modern control theory and has gained increasing attention. A concrete example is a wind park with on-line measurable rotor speeds of the generators as e.g. described in Tien et al. (2016). For gain-scheduled controller synthesis, we take such measurements into account to improve performance, which is in contrast to a classical robust design where these measurements are considered as uncertainties.

On the one hand, gain-scheduling synthesis is often approached in the literature by using parameter-dependent Lyapunov functions (see e.g. de Souza and Trofino (2006) for $\mathcal{H}_2$-performance with state-feedback or Wu and Dong (2005a) for a combination with scalings) or by focusing on scheduling operators which, for example, can be delays as in de Oliveira and Geromel (2004).
On the other hand, there exist scaling approaches that are based on the feedback configuration in Fig. 1 consisting of a linear parametrically-varying (LPV) system $G(\delta)$ and a to-be-designed LPV controller $K(\delta)$ with time-varying parametric scheduling block $\delta = \delta(t)$ which has proven to be an insightful starting point for synthesis (see Packard (1994), Apkarian and Gahinet (1995)). If δ is complex and bounded by $|\delta(t)| \leq 1$ for $t \geq 0$, linear matrix inequalities (LMIs) are used with so-called constant D-scalings in Packard (1994), Apkarian and Gahinet (1995), Wu and Dong (2005b), while a solution for $\delta(t) \in [-1, 1]$ is given in Scorletti and El Ghaoui (1998) with less conservative constant block-diagonal D/G-scalings. Moreover, if δ is restricted to be real with $\delta(t) \geq 0$ for $t \geq 0$, i.e., passive, suitable scalings are provided in Helmersson (1998). Further, the generalization to full-block scalings is considered in Scherer (2000b), while the recent work of Guo and Scherer (2018) is dedicated to structured robust gain-scheduled controller design with $\mathcal{L}_2$-performance based on a suitable scaling factorization for positive definite matrices.

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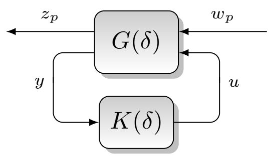

Fig. 1. Feedback-loop for gain-scheduled controller design.

As a first contribution of this paper, we provide a synthesis framework based on scalings for the configuration in Fig. 1 with an $\mathcal{H}_2$-cost criterion imposed on $w_p \to z_p$ and with δ being either passive or complex valued with absolute value bounded by 1. Technically, all existing scaling approaches rely on eliminating the controller parameters. As known from nominal synthesis, this prevents us from considering $\mathcal{H}_2$-design. Therefore, we present for the first time a complete controller parameter transformation for gain-scheduling based on Masubuchi et al. (1998), Scherer et al. (1997).
Furthermore, $\mathcal{H}_2$-synthesis requires to guarantee finiteness of the $\mathcal{H}_2$-norm which, in the related gain-scheduling literature, is often achieved by initially imposing several restrictions for the uncertainty model, controller or the closed-loop system (see e.g. de Souza and Trofino (2006), Scherer (2000b)). As a second contribution, we systematically address this issue by using tailored linear fractional transformations (LFTs) for $G(\delta)$ and $K(\delta)$. In particular, well-posedness of the $\mathcal{H}_2$-norm for the closed-loop requires only an assumption for the direct feedthrough terms of $G(\delta)$ and $K(\delta)$ which amounts to dealing with a structured design problem for partially triangular controller matrices. Such structured $\mathcal{H}_2$-design problems are solved without elimination for block-triangular controller matrices in Voulgaris (2000) based on Youla-parameterization methods, as well as by the direct state-space approaches in Lessard and Lall (2015) or Scherer (2014) relying on coupled algebraic Riccati equations or LMIs, respectively. Technically, the latter approach uses a structured convexifying parameter transformation along with a structured factorization for symmetric, positive definite Lyapunov matrices. For passive δ, a further contribution of this paper

is a new factorization for scaling matrices M satisfying a positive real property, i.e., $M + M^T$ is positive definite.

The paper is organized as follows. After briefly describing the relevant notation, Section 2 introduces the $\mathcal{H}_2$-gain scheduling problem with the proper assumption for the plant/controller LFT to render the $\mathcal{H}_2$-norm finite. The associated synthesis problem is solved in Section 3 for D- and positive real scalings with a controller parameter transformation. Moreover, applications to delayed systems are presented in Section 4.1 and 4.2. As a last contribution, Section 4.3 extends our framework to $\mathcal{H}_2$-synthesis for delayed networked LPV systems which requires to design, in addition to $K(\delta)$, a parametric controller component.

Notation. Let $\mathbb{D}_{\leq}$ be the closed unit disc in $\mathbb{C}$ and $\mathbb{R}_{\geq}$ be the nonnegative real axis. For some matrices $M \in \mathbb{R}^{r\times s}$ and $P \in \mathbb{R}^{r\times r}$ we abbreviate M^TPM by $(*)^TPM$ and $P+P^T$ by $\mathrm{He}(P)$, denote by $\mathrm{tr}(P)$ the trace of P, and call P positive real (PR) if $\mathrm{He}(P) \succ 0$. Matrix entries that are irrelevant or can be inferred by symmetry are indicated by $*$. Further, we drop superscripts specifying partitions and dimensions of matrices if they are clear from the context. We exploit the abbreviation $\mathrm{col}(u_1, u_2) := \begin{pmatrix} u_1^T & u_2^T \end{pmatrix}^T$ for vectors and matrices and denote by I and I_m identity matrices (with m specifying the dimension if not clear from the context). If a transfer matrix $J(s)$ has the realization $C(sI-A)^{-1}B + D$, we express this fact by $J = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Further, $\mathbb{R}(s)$ denotes the set of real-rational functions.

2. PROBLEM FORMULATION

First, we introduce the $\mathcal{H}_2$-gain scheduling problem involving a compact value set $0 \in \mathbf{V}$ that is contained in $\mathbb{D}_{\leq}$ if using D-scalings or in $\mathbb{R}_{\geq}$ if employing positive real scalings. Let us consider Fig. 1 and assume that the LPV system $G(\delta)$ is described for some parameter $\delta \in \mathbf{V}$ by

$$\begin{pmatrix} z_p \\ y \end{pmatrix} = \begin{pmatrix} G_{11}(\delta) & G_{12}(\delta) \\ G_{21}(\delta) & G_{22}(\delta) \end{pmatrix} \begin{pmatrix} w_p \\ u \end{pmatrix} \tag{1}$$

with a performance channel $w_p \to z_p$ and a control channel $u \to y$, as well as parameterized transfer matrices $G_{ij}(\delta)$ in δ whose elements can be expressed as a fraction of two polynomials in δ over the field $\mathbb{R}(s)$ such that the denominator does not vanish at $\delta = 0$. Each $G_{ij}(\delta)$ represents an LPV system compactly written in an input-output description. As we focus on $\mathcal{H}_2$-performance for the uncertain controlled interconnection of Fig. 1, we have to make sure that the direct feedthrough term of the channel $w_p \to z_p$ is zero such that the $\mathcal{H}_2$-norm is finite. To achieve this requirement, let us suppose that the direct feedthrough block of $G_{11}(\delta)$ vanishes. Moreover, to ensure well-posedness of the controller feedback loop, we suppose that the direct feedthrough block of $G_{22}(\delta)$ nominally vanishes, i.e. $G_{22}(0) = 0$. Under these assumptions, we can describe (1) by

$$\begin{pmatrix} \dot{x} \\ \hline z_p \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A(\delta) & B^p(\delta) & B^u(\delta) \\ \hline C^p(\delta) & 0 & D^u(\delta) \\ C^y(\delta) & D^y(\delta) & D(\delta) \end{array}\right) \begin{pmatrix} x \\ \hline w_p \\ u \end{pmatrix} \tag{2}$$

with $D(0) = 0$. Let us emphasize that the zero block for $w_p \to z_p$ reflects the hypothesis on the direct feedthrough block of $G_{11}(\delta)$. By using standard manipulations, this allows for representing such a system as the structured linear fractional representation (LFR)

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \hline z_1 \\ \hline z_2 \\ \hdashline z_p \\ y \end{pmatrix} &= \left(\begin{array}{c|c:cc} A_{11} & A_{12} & B_1^p & B_1 \\ \hline A_{21} & A_{22} & B_2^p & B_2 \\ \hdashline C_1^p & C_2^p & D^p & D_1 \\ C_1 & C_2 & D_2 & D_3 \end{array}\right) \begin{pmatrix} x \\ \hline w_1 \\ \hline w_2 \\ \hdashline w_p \\ u \end{pmatrix}, \qquad w = (\delta I_{r^s})z \\ &= \left(\begin{array}{c|c|c:cc} A_{11} & \bar{A}_{12} & \bar{A}_{13} & B_1^p & B_1 \\ \hline \bar{A}_{21} & \bar{A}_{22} & 0 & 0 & \bar{B}_2 \\ \hline \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & \bar{B}_3^p & \bar{B}_3 \\ \hdashline C_1^p & \bar{C}_2^p & 0 & 0 & D_1 \\ C_1 & \bar{C}_2 & \bar{C}_3 & D_2 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w_1 \\ \hline w_2 \\ \hdashline w_p \\ u \end{pmatrix} \end{aligned} \tag{3}$$

with a triangular matrix A_{22} and structured B_2^p, C_2^p, as well as $D^p = 0$ and $D_3 = 0$; here $w \to z$ is the uncertainty channel with $w = \mathrm{col}(w_1, w_2)$, $z = \mathrm{col}(z_1, z_2)$ partitioned as $r^s = r_1^s + r_2^s$, and $A_{11} \in \mathbb{R}^{n^s\times n^s}$, $B_1 \in \mathbb{R}^{n^s\times m}$, $C_1 \in \mathbb{R}^{k\times n^s}$ are unstructured matrices.

Let us denote by $\mathbf{\Delta}$ the set of all admissible time-varying parametric uncertainties encompassing all continuous curves $\delta : [0, \infty) \to \mathbf{V}$; we allow for complex valued uncertainties to handle a larger class of interesting problems as e.g. delayed systems in Section 4.1. The LFR in (3) for time-varying $\delta \in \mathbf{\Delta}$ is then the precise mathematical plant description for us to work with.

Moreover, as a further step to guarantee finiteness of the $\mathcal{H}_2$-norm, we only search for gain-scheduling controllers $K(\delta)$ with vanishing direct feedthrough term, i.e. $K(\delta)$ is described by

$$\begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A^c(\delta) & B^c(\delta) \\ C^c(\delta) & 0 \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix}. \tag{4}$$

In view of the structural similarities to (2), the corresponding controller LFR is taken to be

$$\begin{aligned} \begin{pmatrix} \dot{x}_c \\ \hline z_{c,1} \\ z_{c,2} \\ \hdashline u \end{pmatrix} &= \left(\begin{array}{c|c:c} A_{11}^c & A_{12}^c & B_1^c \\ \hline A_{21}^c & A_{22}^c & B_2^c \\ \hdashline C_1^c & C_2^c & D^c \end{array}\right) \begin{pmatrix} x_c \\ \hline w_{c,1} \\ w_{c,2} \\ \hdashline y \end{pmatrix}, \qquad w_c = (\delta I_{r^c})z_c \\ &= \left(\begin{array}{c|cc:c} A_{11}^c & \bar{A}_{12}^c & \bar{A}_{13}^c & B_1^c \\ \hline \bar{A}_{21}^c & \bar{A}_{22}^c & 0 & 0 \\ \bar{A}_{31}^c & \bar{A}_{32}^c & \bar{A}_{33}^c & \bar{B}_3^c \\ \hdashline C_1^c & \bar{C}_2^c & 0 & 0 \end{array}\right) \begin{pmatrix} x_c \\ \hline w_{c,1} \\ w_{c,2} \\ \hdashline y \end{pmatrix} \end{aligned} \tag{5}$$

with $\delta \in \mathbf{\Delta}$ and matrices $\left(\begin{array}{c:c} A_{22}^c & B_2^c \\ \hdashline C_2^c & D^c \end{array}\right)$ inheriting the sparsity pattern of $\left(\begin{array}{c:c} A_{22} & B_2^p \\ \hdashline C_2^p & D^p \end{array}\right)$ in (3), while the scheduling channel $w_c \to z_c$ involves $w_c = \mathrm{col}(w_{c,1}, w_{c,2})$, $z_c = \mathrm{col}(z_{c,1}, z_{c,2})$ of partition $r^c = r_1^c + r_2^c$. Further, the choice of r^c and the dimension n^c for the unstructured unknowns $A_{11}^c \in \mathbb{R}^{n^c\times n^c}$, $B_1^c \in \mathbb{R}^{n^c\times k}$, $C_1^c \in \mathbb{R}^{m\times n^c}$ are part of the design problem. In the sequel, we also abbreviate $n := n^s + n^c$ and $r := r^s + r^c$.

Due to $D_3 = 0$, the controlled interconnection of (3) and (5) can be routinely expressed for $\delta \in \mathbf{\Delta}$ by

$$\begin{pmatrix} \dot{x}_e \\ \hdashline z_e \\ \hdashline z_p \end{pmatrix} \left(\begin{array}{c:c:c} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{B}_1 \\ \hdashline \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{B}_2 \\ \hdashline \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{D} \end{array}\right) \begin{pmatrix} x_e \\ \hdashline w_e \\ \hdashline w_p \end{pmatrix}, \qquad w_e = (\delta I_r)z_e \tag{6}$$

with the extended signals $x_e := \mathrm{col}(x, x_c)$, $z_e := \mathrm{col}(z, z_c)$, $w_e := \mathrm{col}(w, w_c)$ and with closed-loop matrices given as

$$\left(\begin{array}{c|c|c} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{B}_1 \\ \hline \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{D} \end{array}\right) = \left(\begin{array}{cc|cc|c} A_{11} & 0 & A_{12} & 0 & B_1^p \\ 0 & 0 & 0 & 0 & 0 \\ \hline A_{21} & 0 & A_{22} & 0 & B_2^p \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_1^p & 0 & C_2^p & 0 & D^p \end{array}\right) +$$

$$+ \left(\begin{array}{c:c:c} 0 & 0 & B_1 \\ I_{n^c} & 0 & 0 \\ \hline 0 & 0 & B_2 \\ 0 & I_{r^c} & 0 \\ \hline 0 & 0 & D_1 \end{array}\right) \left(\begin{array}{c:c:c} A_{11}^c & A_{12}^c & B_1^c \\ \hdashline A_{21}^c & A_{22}^c & B_2^c \\ \hdashline C_1^c & C_2^c & D^c \end{array}\right) \left(\begin{array}{cc|cc|c} 0 & I_{n^c} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & I_{r^c} & 0 \\ \hdashline C_1 & 0 & C_2 & 0 & D_2 \end{array}\right). \tag{7}$$

Definition 1. The controlled interconnection (6) is *well-posed* if $I - v\mathcal{A}_{22}$ is non-singular for all $v \in \mathbf{V}$. It is called *stable* if there exist constants K and $\alpha > 0$ such that every solution of (6) for $w_p = 0$ and for any $\delta \in \mathbf{\Delta}$ satisfies

$$\|x_e(t)\| \leq K e^{-\alpha(t-t_0)} \|x_e(t_0)\| \quad \text{for all} \quad t \geq t_0 \geq 0.$$

If (6) is well-posed, we can close the loop with the scheduling block δI_r to obtain the uncertain closed-loop description which, in the sequel, is denoted by $\left(\begin{smallmatrix} \dot{x}_e \\ z_p \end{smallmatrix}\right) = \left(\begin{smallmatrix} \boldsymbol{\mathcal{A}}(\delta) & \boldsymbol{\mathcal{B}}(\delta) \\ \boldsymbol{\mathcal{C}}(\delta) & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_e \\ w_p \end{smallmatrix}\right)$. As desired, the direct feedthrough block of the performance channel of this system is identically zero. This also results from interconnecting (2) with (4) and is hence only achieved due to the specific choice of the LFRs (3), (5). The $\mathcal{H}_2$*-gain-scheduling problem* thus translates into a structured design problem as follows.

Problem 2. For a given bound $\gamma > 0$, determine a controller $K(\delta)$ structured as in (5) such that

(G1) the controlled interconnection (6) is well-posed and stable,

(G2) the squared $\mathcal{H}_2$-norm of $w_p \to z_p$ for linear time-varying systems (with the stochastic interpretation as in Paganini and Feron (2000)) is smaller than γ for $x_e(0) = 0$ and for all $\delta \in \mathbf{\Delta}$.

In order to render this problem computationally tractable, we introduce the following class of D-scalings $\mathcal{S}$ and of positive real scalings $\mathcal{R}$:

$$\mathcal{S} := \left\{ \left(\begin{smallmatrix} -Q & 0 \\ 0 & Q \end{smallmatrix}\right) \,\middle|\, Q \succ 0 \right\},\ \mathcal{R} := \left\{ \left(\begin{smallmatrix} 0 & Q \\ Q^T & 0 \end{smallmatrix}\right) \,\middle|\, \mathrm{He}(Q) \succ 0 \right\}.$$

Note that the blocks Q in $\mathcal{S}$ have to be symmetric, while those in $\mathcal{R}$ can be unstructured. As a step towards synthesis, the following standard result with matrix inequalities can then be derived after applying the so-called full-block S-procedure (c.f. Scherer (2000b)).

Theorem 3. Let $\mathcal{P} = \mathcal{S}$ or $\mathcal{P} = \mathcal{R}$. The controller $K(\delta)$ structured as in (5) achieves (G1), (G2) if there exist $\mathcal{X}_1 \succ 0$, $Z \succ 0$ with $\mathrm{tr}(Z) < 1$ as well as $P_1, P_2 \in \mathcal{P}$ such that

$$(*)^T \left(\begin{array}{cc:c:c} 0 & \mathcal{X}_1 & 0 & 0 \\ \mathcal{X}_1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & P_1 & 0 \\ \hdashline 0 & 0 & 0 & P_p \end{array}\right) \left(\begin{array}{ccc} I_n & 0 & 0 \\ \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{B}_1 \\ \hdashline 0 & I_r & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{B}_2 \\ \hdashline 0 & 0 & I \\ \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{D} \end{array}\right) \prec 0,$$

$$(*)^T \left(\begin{array}{c:c:c} -\mathcal{X}_1 & 0 & 0 \\ \hdashline 0 & P_2 & 0 \\ \hdashline 0 & 0 & Z^{-1} \end{array}\right) \left(\begin{array}{cc} I_n & 0 \\ \hdashline 0 & I_r \\ \mathcal{A}_{21} & \mathcal{A}_{22} \\ \hdashline \mathcal{C}_1 & \mathcal{C}_2 \end{array}\right) \prec 0 \tag{8}$$

hold for (6) and $P_p = \left(\begin{smallmatrix} -\gamma I & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

3. SYNTHESIS

3.1 Synthesis for Positive Real Scalings

In order to solve the $\mathcal{H}_2$-design Problem 2 by using positive real scalings, we aim to match (8) with convex synthesis constraints in terms of LMIs. An elimination of the controller matrices, as e.g. applied in Scherer (2000b), is not possible since some of these matrices affect both inequalities in (8). As a remedy, we use a novel structured factorizing for positive real scalings combined with a suitable convexifying controller parameter transformation. As introduced in the notations and in what follows, we call a positive real matrix shortly PR. To formulate the corresponding main result, we introduce the associated variables that consist of the symmetric unknowns X_1 and Y_1 of dimension $n^s \times n^s$. Moreover, as an essential new ingredient for the structured controller design with positive real scalings, we take the rectangular variables $X_2, Y_2 \in \mathbb{R}^{r^s \times (r^s + r^s)}$ which are partitioned according to

$$X_2 := \left(X_{22} \ X_{23} \right) \ \text{ and } \ Y_2 := \left(Y_{21} \ Y_{22} \right) \tag{9}$$

with PR matrices X_{23}, Y_{21} and structured blocks

$$X_{22} := \begin{pmatrix} \hat{X}_2 & \hat{Z}_2 \\ 0 & I_{r_2^s} \end{pmatrix}, \quad Y_{22} := \begin{pmatrix} I_{r_1^s} & 0 \\ \hat{Z}_1 & \hat{Y}_2 \end{pmatrix} \tag{10}$$

which are composed of the PR matrices $\hat{X}_2, \hat{Y}_2$ of dimension $r_1^s \times r_1^s$, $r_2^s \times r_2^s$ and the unstructured real matrix variables $\hat{Z}_1, \hat{Z}_2$ of dimension $r_2^s \times r_1^s$, $r_1^s \times r_2^s$. In addition, let us also define the partially structured block matrix

$$\left(\begin{array}{c:c:c} K_{11} & K_{12} & L_1 \\ \hdashline K_{21} & K_{22} & L_2 \\ \hdashline M_1 & M_2 & N \end{array}\right) := \left(\begin{array}{c:cc:c} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} & \bar{L}_1 \\ \hdashline \bar{K}_{21} & \bar{K}_{22} & X_{22}^T A_{22} Y_{22} & 0 \\ \bar{K}_{31} & \bar{K}_{32} & \bar{K}_{33} & \bar{L}_3 \\ \hdashline \bar{M}_1 & \bar{M}_2 & 0 & 0 \end{array}\right) \tag{11}$$

of dimension $(n^s + r^s + r^s + m) \times (n^s + r^s + r^s + k)$ comprising the rectangular variables $\bar{K}_{ij}$, $\bar{L}_i$, $\bar{M}_j$, as well as $X_{22}^T A_{22} Y_{22} = \left(\begin{smallmatrix} \hat{X}_2^T \bar{A}_{22} & 0 \\ \bar{A}_{32} + \bar{A}_{33}\hat{Z}_1 + \hat{Z}_2^T \bar{A}_{22} & \bar{A}_{33}\hat{Y}_2 \end{smallmatrix}\right)$ with the unknowns $\hat{X}_2, \hat{Y}_2, \hat{Z}_1, \hat{Z}_2$ from (10).

If both multipliers P_1 and P_2 are taken to be identical in (8), we obtain the following result for synthesis.

Theorem 4. Let P_p and $Z \succ 0$ with $\mathrm{tr}(Z) < 1$ be given as in Theorem 3. There exists a controller (5) such that the inequalities (8) are satisfied for (6) with some $\mathcal{X}_1 \succ 0$ and $P_1 = P_2 \in \mathcal{R}$ iff there exist symmetric $X_1, Y_1 \in \mathbb{R}^{n^s \times n^s}$, structured $X_2, Y_2 \in \mathbb{R}^{r^s \times (r^s + r^s)}$ from (9), (10) as well as K_{ij}, L_i, M_j from (11) such that the inequalities

$$(*)^T \left(\begin{array}{cc:cc:c} 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & P_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{B}_1 \\ \hdashline 0 & I & 0 \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{B}_2 \\ \hdashline 0 & 0 & I \\ \boldsymbol{C}_1 & \boldsymbol{C}_2 & \boldsymbol{D} \end{array}\right) \prec 0 \tag{12}$$

and

$$(*)^T \left(\begin{array}{c:cc:c} -\boldsymbol{X}_1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ \hdashline 0 & 0 & 0 & Z^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \hdashline 0 & I \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \\ \hdashline \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{array}\right) \prec 0 \tag{13}$$

as well as

$$\mathrm{He}(Y_{21}) \succ 0, \quad \mathrm{He}(X_{23}) \succ 0, \quad \mathrm{He}\begin{pmatrix} \hat{X}_2 & \hat{Z}_2 \\ \hat{Z}_1 & \hat{Y}_2 \end{pmatrix} \succ 0 \tag{14}$$

are fulfilled after inserting for $i, j = 1, 2$ the blocks

$$\boldsymbol{X}_1 := \begin{pmatrix} Y_1 & I_{n^s} \\ I_{n^s} & X_1 \end{pmatrix},$$

$$\left(\begin{array}{c|c} \boldsymbol{A}_{ij} & \boldsymbol{B}_i \\ \hline \boldsymbol{C}_j & \boldsymbol{D} \end{array}\right) := \left(\begin{array}{cc|c} A_{ij}Y_j & A_{ij} & B_i^p \\ 0 & X_i^T A_{ij} & X_i^T B_i^p \\ \hline C_j^p Y_j & C_j^p & D^p \end{array}\right) + \\ + \left(\begin{array}{cc} 0 & B_i \\ I & 0 \\ \hline 0 & D_1 \end{array}\right) \begin{pmatrix} K_{ij} & L_i \\ M_j & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_j & D_2 \end{array}\right). \tag{15}$$

Despite the fact that D^p and N actually vanish, we still depict these matrices in (15) as a preparation for Section 4.1. If applying the Schur complement to (13), we arrive at a standard LMI test. The following proof is constructive, i.e., if (12), (13), (14) are satisfied for (15), a controller can be computed to render (8) valid for (6). Moreover, the construction can be performed such that the controller McMillan degree is at most n^s, whereas the degree r^c of the scheduling channel $w_c = (\delta I_{r^c})z_c$ is at most $2r^s$.

Proof. *Necessity.* Let (8) being satisfied for (6), $\mathcal{X}_1 \succ 0$ as well as for $P_1 = P_2 \in \mathcal{R}$. By definition of $\mathcal{R}$ we can express $P_1 = P_2 = \begin{pmatrix} 0 & \mathcal{X}_2 \\ \mathcal{X}_2^T & 0 \end{pmatrix}$ with $\mathcal{X}_2$ of dimension $r \times r$ being PR. Hence $\mathcal{X}_1$ and $\mathcal{X}_2$ are invertible. Moreover, w.l.o.g. we can assume that $n^c \geq n^s$ which leads for $i = 1$ to the factorizations (see Scherer et al. (1997))

$$\mathcal{X}_i \mathcal{Y}_i = \mathcal{Z}_i \quad \text{with} \quad \mathcal{Y}_i := \begin{pmatrix} Y_i & I \\ V_i & 0 \end{pmatrix}, \ \mathcal{Z}_i := \begin{pmatrix} I & X_i \\ 0 & U_i \end{pmatrix} \tag{16}$$

such that $\mathcal{Y}_1$ has full column rank. Further, w.l.o.g. let us assume that $r_1^c \geq r^s$ and $r_2^c \geq r^s$. As a key step, we clarify in Appendix A that $\mathcal{X}_2$ can be factorized as in (16) with $\mathcal{Y}_2$ of full column rank, and such that V_2 and U_2 are lower and upper block-triangular matrices, respectively, in the partition $(r_1^c + r_2^c) \times (r^s + r^s)$, while Y_2 and X_2 are structured as in (9), (10). By applying congruence transformations with $\mathcal{Y}_i$ along with the corresponding factorization (16) for $i = 1, 2$, the inequalities in (8) and $\mathrm{He}(\mathcal{X}_2) \succ 0$ lead to $\mathrm{He}(\mathcal{Z}_2^T \mathcal{Y}_2) \succ 0$ and

$$(*)^T \left(\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & P_p \end{array}\right) \left(\begin{array}{ccc} I & 0 & 0 \\ \mathcal{Z}_1^T \mathcal{A}_{11} \mathcal{Y}_1 & \mathcal{Z}_1^T \mathcal{A}_{12} \mathcal{Y}_2 & \mathcal{Z}_1^T \mathcal{B}_1 \\ \hline 0 & I & 0 \\ \mathcal{Z}_2^T \mathcal{A}_{21} \mathcal{Y}_1 & \mathcal{Z}_2^T \mathcal{A}_{22} \mathcal{Y}_2 & \mathcal{Z}_2^T \mathcal{B}_2 \\ \hline 0 & 0 & I \\ \mathcal{C}_1 \mathcal{Y}_1 & \mathcal{C}_2 \mathcal{Y}_2 & \mathcal{D} \end{array}\right) \prec 0,$$

$$(*)^T \left(\begin{array}{c|cc|c} -\mathcal{Z}_1^T \mathcal{Y}_1 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & Z^{-1} \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \hline 0 & I \\ \mathcal{Z}_2^T \mathcal{A}_{21} \mathcal{Y}_1 & \mathcal{Z}_2^T \mathcal{A}_{22} \mathcal{Y}_2 \\ \hline \mathcal{C}_1 \mathcal{Y}_1 & \mathcal{C}_2 \mathcal{Y}_2 \end{array}\right) \prec 0. \tag{17}$$

Now, by definition of V_2, U_2 in Appendix A, let us express

$$V_2^T U_2 =: \begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{pmatrix} \tag{18}$$

such that $\mathrm{He}(\mathcal{Z}_2^T \mathcal{Y}_2) \succ 0$ explicitly reads as

$$\begin{pmatrix} \mathrm{He}(Y_{21}) & Y_{22} + Y_{21}^T X_{22} + W_{11} & I + Y_{21}^T X_{23} + W_{12} \\ * & \mathrm{He}(Y_{22}^T X_{22}) & X_{22}^T + Y_{22}^T X_{23} + W_{23} \\ * & * & \mathrm{He}(X_{23}) \end{pmatrix} \succ 0. \tag{19}$$

Moreover, $\mathrm{He}(Y_{22}^T X_{22}) = \mathrm{He}\begin{pmatrix} \hat{X}_2 & \hat{Z}_2 \\ \hat{Z}_1 & \hat{Y}_2 \end{pmatrix}$ is true by definition (10) and, thus, (14) follows from (19). Further, we extract

$$\mathcal{Z}_1^T \mathcal{Y}_1 = \begin{pmatrix} Y_1 & I_{n^s} \\ I_{n^s} & X_1 \end{pmatrix} \tag{20}$$

which relates to the usual coupling condition and corresponds to $\boldsymbol{X}_1$ in (15). We emphasize that X_1, Y_1 are symmetric matrices. In order to finish the necessity proof, let us zoom into the outer factors of the inequalities in (17). Simple calculations reveal that

$$\left(\begin{array}{c|c} \mathcal{Z}_i^T \mathcal{A}_{ij} \mathcal{Y}_j & \mathcal{Z}_i^T \mathcal{B}_i \\ \hline \mathcal{C}_j \mathcal{Y}_j & \mathcal{D} \end{array}\right) = \left(\begin{array}{cc|c} A_{ij}Y_j & A_{ij} & B_i^p \\ 0 & X_i^T A_{ij} & X_i^T B_i^p \\ \hline C_j^p Y_j & C_j^p & D^p \end{array}\right) + \\ + \left(\begin{array}{cc} 0 & B_i \\ I & 0 \\ \hline 0 & D_1 \end{array}\right) \begin{pmatrix} K_{ij} & L_i \\ M_j & N \end{pmatrix} \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_j & D_2 \end{array}\right) \tag{21}$$

for $i, j = 1, 2$ after substituting

$$\left(\begin{array}{cc|c} K_{11} & K_{12} & L_1 \\ K_{21} & K_{22} & L_2 \\ \hline M_1 & M_2 & N \end{array}\right) := \left(\begin{array}{cc|c} X_1^T A_{11} Y_1 & X_1^T A_{12} Y_2 & 0 \\ X_2^T A_{21} Y_1 & X_2^T A_{22} Y_2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) + \\ + \left(\begin{array}{cc|c} U_1^T & 0 & X_1^T B_1 \\ 0 & U_2^T & X_2^T B_2 \\ \hline 0 & 0 & I_m \end{array}\right) \left(\begin{array}{cc|c} A_{11}^c & A_{12}^c & B_1^c \\ A_{21}^c & A_{22}^c & B_2^c \\ \hline C_1^c & C_2^c & D^c \end{array}\right) \left(\begin{array}{cc|c} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ \hline C_1 Y_1 & C_2 Y_2 & I_k \end{array}\right). \tag{22}$$

Since relevant in Section 4.1, we also display $D^p = 0$, $D^c = 0$, $N = 0$ and $\mathcal{D} = 0$ in the above formula. Due to the triangular structure of U_2, V_2 and the structure of the controller matrices with $D^c = 0$, we observe that

$$\left(\begin{array}{cc|c} * & 0 & 0 \\ * & * & \bar{L}_3 \\ \hline \bar{M}_2 & 0 & 0 \end{array}\right) := \left(\begin{array}{c|c} \bar{A} & L_2 \\ \hline M_2 & 0 \end{array}\right) := \left(\begin{array}{c|c} U_2^T A_{22}^c V_2 & U_2^T B_2^c \\ \hline C_2^c V_2 & 0 \end{array}\right) \tag{23}$$

are sparse matrices, which allows to infer from (22) that

$$\begin{aligned} K_{22} &= X_2^T A_{22} Y_2 + \bar{A} + L_2 C_2 Y_2 + X_2^T B_2 M_2 \\ &= \begin{pmatrix} * & X_{22}^T A_{22} Y_{22} \\ * & * \end{pmatrix} + \bar{A} + \begin{pmatrix} 0 \\ \bar{L}_3 \end{pmatrix} C_2 Y_2 + X_2^T B_2 \begin{pmatrix} \bar{M}_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & X_{22}^T A_{22} Y_{22} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}. \end{aligned} \tag{24}$$

Therefore, by defining $\begin{pmatrix} \bar{K}_{22} & 0 \\ \bar{K}_{32} & \bar{K}_{33} \end{pmatrix} := K_{22} - \begin{pmatrix} 0 & X_{22}^T A_{22} Y_{22} \\ 0 & 0 \end{pmatrix}$, the inequality (17) implies (12) and (13) for (15).

Sufficiency. Suppose that the inequalities in Theorem 4 are feasible. This means that there exist symmetric X_1, Y_1, structured variables X_2, Y_2 from (9), (10), as well as K_{ij}, L_i, M_j from (11) such that (12)-(14) hold after inserting (15). Let us now set $U_1 := I_{n^s}$, $V_1 := I_{n^s} - X_1^T Y_1$ to define $\mathcal{Y}_1$, $\mathcal{Z}_1$ by (16). Due (13) we get $\boldsymbol{X}_1 \succ 0$ and hence the choice of U_1, V_1 shows that (20) is satisfied. Moreover, (14) implies that Y_{21}, X_{23}, $\hat{X}_2$ and $\hat{Y}_2$ are invertible. By definition, we note that X_{22}, Y_{22} are invertible, too. In view of the strict right upper part of (19), let us choose

$$V_2 := -\begin{pmatrix} Y_{21} & 0 \\ 0 & Y_{22} \end{pmatrix}, \ U_2 := \begin{pmatrix} X_{22} & X_{23} \\ 0 & X_{23} \end{pmatrix} - V_2^{-T} \begin{pmatrix} Y_{22} & I \\ 0 & X_{22}^T \end{pmatrix}$$

with a triangular structure to infer by inspection that

$$-V_2^T U_2 = \begin{pmatrix} Y_{21} & 0 \\ 0 & Y_{22} \end{pmatrix}^T \begin{pmatrix} X_{22} & X_{23} \\ 0 & X_{23} \end{pmatrix} + \begin{pmatrix} Y_{22} & I \\ 0 & X_{22}^T \end{pmatrix}.$$

In particular, this shows with (18) and (14) that (19) is true. Hence, by defining $\mathcal{Y}_2$, $\mathcal{Z}_2$ through (16), this implies $\mathrm{He}(\mathcal{Z}_2^T \mathcal{Y}_2) \succ 0$. We note that U_1, V_1 and V_2 are invertible. Moreover, U_2 is also invertible since its diagonal blocks are invertible. Exemplarily, let us show this for $X_{22} + Y_{21}^{-T} Y_{22}$: We infer from (19) that $\mathrm{He}(Y_{22}^T X_{22}) \succ 0$ which, by applying a congruence transformation with Y_{22}^{-1}, shows that $X_{22} Y_{22}^{-1}$ is PR. Further, Y_{21}^{-1} is PR by (14) which implies that $X_{22}Y_{22}^{-1} + Y_{21}^{-T}$ is PR and thus, $X_{22} + Y_{21}^{-T} Y_{22}$ is invertible. Hence $\mathcal{Y}_i$ and $\mathcal{Z}_i$ are invertible which allows to define $\mathcal{X}_i := \mathcal{Z}_i \mathcal{Y}_i^{-1}$ such that (16) is satisfied for $i = 1, 2$. Furthermore, in view of (24), let us define

$$\bar{A} := \begin{pmatrix} \bar{K}_{22} & X_{22}^T A_{22} Y_{22} \\ \bar{K}_{32} & \bar{K}_{33} \end{pmatrix} - X_2^T A_{22} Y_2 - L_2 C_2 Y_2 - X_2^T B_2 M_2$$

to infer that (24) is true for the full matrices. Moreover, since U_i, V_i are invertible, we can solve (22) for the controller blocks A_{ij}^c, B_i^c, C_j^c which shows that (21) is true for the closed-loop matrices (6). In particular, we note that $\bar{A}$ is lower block-triangular by definition. Thus (23) is satisfied which shows that A_{22}^c, B_2^c and C_2^c have the desired sparsity pattern as in (5). Hence the inequalities (12), (13) with (15) are equivalent to (17). By using (16) and congruence transformations with $\mathcal{Y}_i^{-1}$ for $i = 1, 2$, this leads back to (8) for (6). ∎

Remark 5. For quadratic performance, let $P_p := \left(\begin{smallmatrix} Q_p & * \\ S_p & R_p \end{smallmatrix}\right)$ with $R_p \succcurlyeq 0$ denote the associated performance index. A sufficient synthesis inequality for the corresponding gain-scheduling problem with positive real scalings is then given by (12) and (14) after replacing P_γ by P_p and inserting (15). In particular, the parameter transformation approach in Theorem 4 allows to solve the gain-scheduling problem for $\mathcal{H}_\infty$-performance as well as for multiple objectives.

Remark 6. Since the unstructured K_{11}, K_{12}, L_1 affect only (12) and not (13), they can be partially eliminated with the standard elimination lemma (e.g. Lemma A.3 in Scherer (2000b)). This allows to reduce the computational complexity by decreasing the number of the LMI variables.

3.2 Synthesis for D-scalings

Analogously to Theorem 4, we now sketch how to derive a synthesis result for D-scalings. If the analysis inequalities (8) are fulfilled for (6), $\mathcal{X}_1 \succ 0$, $Z \succ 0$ with $\mathrm{tr}(Z) < 1$ as well as for $P_1 = P_2 \in \mathcal{S}$, we can express $P_1 = P_2 = \left(\begin{smallmatrix} -\mathcal{X}_2 & 0 \\ 0 & \mathcal{X}_2 \end{smallmatrix}\right)$ with $\mathcal{X}_2 \succ 0$ of dimension $r \times r$. If treating $\mathcal{X}_2$ in the same fashion as the Lyapunov variable $\mathcal{X}_1$ (c.f. Guo and Scherer (2018)), similar arguments can be applied as before by taking rectangular variables $X_2, Y_2 \in \mathbb{R}^{r^s \times (r^s + r^s)}$ partitioned according to (9) with symmetric X_{23}, Y_{21}, structured blocks

$$X_{22} := \begin{pmatrix} \hat{X}_2 & \hat{Z}_2^T \\ 0 & I_{r_2^s} \end{pmatrix} \text{ and } Y_{22} := \begin{pmatrix} I_{r_1^s} & 0 \\ -\hat{Z}_2 & \hat{Y}_2 \end{pmatrix} \tag{25}$$

with symmetric $\hat{X}_2 \in \mathbb{R}^{r_1^s \times r_1^s}$, $\hat{Y}_2 \in \mathbb{R}^{r_2^s \times r_2^s}$ and a rectangular $\hat{Z}_2 \in \mathbb{R}^{r_2^s \times r_1^s}$. After applying the Schur complement, the corresponding synthesis result for D-scalings can then be compactly formulated in the same fashion as Theorem 4 by replacing the inequalities (12)-(14) through the LMIs

$$\begin{pmatrix} \mathrm{He}(\boldsymbol{A}_{11}) & * & * & * \\ \boldsymbol{A}_{12}^T & -\boldsymbol{X}_2 & * & * \\ \boldsymbol{B}_1^T & 0 & -\gamma I & * \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{B}_2 & -\boldsymbol{X}_2 \end{pmatrix} \prec 0, \begin{pmatrix} -\boldsymbol{X}_1 & * & * & * \\ 0 & -\boldsymbol{X}_2 & * & * \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & -\boldsymbol{X}_2 & * \\ \boldsymbol{C}_1 & \boldsymbol{C}_2 & 0 & -Z \end{pmatrix} \prec 0 \tag{26}$$

with the function blocks from (15) and with

$$\boldsymbol{X}_2 := \begin{pmatrix} Y_{21} & Y_{22} & I \\ Y_{22}^T & Y_{22}^T X_{22} & X_{22}^T \\ I & X_{22} & X_{23} \end{pmatrix}.$$

If (26) are feasible, one can construct a controller with McMillan degree of at most n^s and r^c bounded by $2r^s$.

4. APPLICATIONS

4.1 Delayed systems

A nice application of our framework is the design of operator-scheduled controllers. As an example, let us consider the scenario in de Oliveira and Geromel (2004). We denote by d_τ the standard delay operator of τ seconds in continuous-time mapping $\hat{u}(.)$ into $\hat{y}(t) = \hat{u}(t - \tau)$ if $t \geq \tau$ and into $\hat{y}(t) = 0$ if $0 \leq t < \tau$. The plant and controller in de Oliveira and Geromel (2004) can then be expressed as

$$\begin{pmatrix} \dot{x} \\ z_p \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A_{00} + A_{01} d_\tau & B^p & B \\ \hline C_{00}^p + C_{01}^p d_\tau & D^p & D_1 \\ C_{00}^y + C_{01}^y d_\tau & D_2 & 0 \end{array}\right) \begin{pmatrix} x \\ w_p \\ u \end{pmatrix} \tag{27}$$

$$\text{and} \quad \begin{pmatrix} \dot{x}_c \\ u \end{pmatrix} = \begin{pmatrix} A_{00}^c + A_{01}^c d_\tau & B^c \\ C_{00}^c + C_{01}^c d_\tau & D^c \end{pmatrix} \begin{pmatrix} x_c \\ y \end{pmatrix},$$

respectively, with initial conditions $x(t) = 0$ and $x_c(t) = 0$ for $t \in [-\tau, 0]$. The descriptions in (27) result from

$$\begin{pmatrix} \dot{x} \\ \hline z_1 \\ \hline z_2 \\ \hdashline z_p \\ y \end{pmatrix} = \left(\begin{array}{c|c|c:cc} A_{11} & \bar{A}_{12} & \bar{A}_{13} & B_1^p & B_1 \\ \hline \bar{A}_{21} & \bar{A}_{22} & 0 & 0 & 0 \\ \hline \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & 0 & 0 \\ \hdashline C_1^p & \bar{C}_2^p & 0 & D^p & D_1 \\ C_1 & \bar{C}_2 & 0 & D_2 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w_1 \\ \hline w_2 \\ \hdashline w_p \\ u \end{pmatrix}, \; w = (d_\tau I_{r^s}) z$$

$$\begin{pmatrix} \dot{x}_c \\ \hline z_{c,1} \\ z_{c,2} \\ \hdashline u \end{pmatrix} = \left(\begin{array}{c|cc:c} A_{11}^c & \bar{A}_{12}^c & \bar{A}_{13}^c & B_1^c \\ \hline \bar{A}_{21}^c & \bar{A}_{22}^c & 0 & 0 \\ \bar{A}_{31}^c & \bar{A}_{32}^c & \bar{A}_{33}^c & 0 \\ \hdashline C_1^c & \bar{C}_2^c & 0 & D^c \end{array}\right) \begin{pmatrix} x_c \\ \hline w_{c,1} \\ w_{c,2} \\ \hdashline y \end{pmatrix}, \; w_c = (d_\tau I_{r^c}) z_c \tag{28}$$

if $w_2 \to z_2$, $w_{c,2} \to z_{c,2}$ are empty. Note that (28) is a modification of (3) and (5) with $D^p \neq 0$, $D^c \neq 0$, $\bar{C}_3 = 0$

$$\text{and} \quad B_2^p = 0, \quad B_2 = 0, \quad B_2^c = 0. \tag{29}$$

Further, if $\| \left(\begin{smallmatrix} \bar{A}_{22} & 0 \\ \bar{A}_{32} & \bar{A}_{33} \end{smallmatrix}\right) \| < 1$, well-posedness of the plant LFR follows from a small-gain argument for d_τ, while the delay variations in time are covered by using D-scalings (c.f. Megretski and Rantzer (1997)). Contrary to Section 2, the analysis conditions in de Oliveira et al. (2002) are based on a deterministic interpretation of the $\mathcal{H}_2$-norm. If (6) is the closed-loop system for (28), these conditions are a special case of [Veenman et al. (2016), Corollary 9] for D-scalings. In our setting, this amounts to replacing (8) for $P_1 = P_2 = \left(\begin{smallmatrix} -\mathcal{X}_2 & 0 \\ 0 & \mathcal{X}_2 \end{smallmatrix}\right) \in \mathcal{S}$, by $\mathrm{tr}(Z) < \gamma$ and

$$\left(\begin{array}{ccc:c} \mathrm{He}(\mathcal{X}_1 \mathcal{A}_{11}) & * & * & * \\ \mathcal{A}_{12}^T \mathcal{X}_1 & -\mathcal{X}_2 & * & * \\ \mathcal{C}_1 & \mathcal{C}_2 & -I & * \\ \hdashline \mathcal{X}_2 \mathcal{A}_{21} & \mathcal{X}_2 \mathcal{A}_{22} & 0 & -\mathcal{X}_2 \end{array}\right) \prec 0, \begin{pmatrix} Z & \mathcal{B}_1^T \mathcal{X}_1 \\ \mathcal{X}_1 \mathcal{B}_1 & \mathcal{X}_1 \end{pmatrix} \succ 0 \,.$$

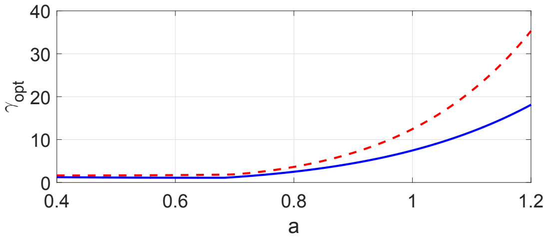

Fig. 2. Optimal bounds γ_{opt} for (syst) with different $a \in [0.4, 1.2]$. The results are given for the unstructured (full blue) and the structured design (dashed red).

Note that this requires to guarantee $\mathcal{B}_2 = 0$, $\mathcal{D} = 0$ in (6), which is achieved due to (29) and by adjoining the equality constraint $\mathcal{D} = D^p + D_1 D^c D_2 = 0$ to the analysis and synthesis conditions. Exactly the same procedure as in Section 3 leads to LMIs in terms of (11) for $\bar{L}_3 = 0$ and $M_2 = (\bar{M}_2 \; D^c C_2)$. Further, convexification in the structured controller parameter transformation (22) is achieved since $\bar{C}_3 = 0$ implies the crucial relation $C_2 Y_{22} = C_2$; for reasons of space, we drop further details.

4.2 A Numerical Example

Based on the Matlab Robust Control Toolbox, let us present a short academic example with (28) for

$$\left(\begin{array}{c|c|c:cc} A_{11} & \bar{A}_{12} & \bar{A}_{13} & B_1^p & B_1 \\ \hline \bar{A}_{21} & \bar{A}_{22} & 0 & 0 & 0 \\ \hline \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & 0 & 0 \\ \hdashline C_1^p & \bar{C}_2^p & 0 & D^p & D_1 \\ C_1 & \bar{C}_2 & 0 & D_2 & 0 \end{array}\right) := \left(\begin{array}{cc|cc|c:cc} 0 & 1 & a & 2.4a & a & 1 & 1 \\ 0 & -3 & 2.4a & -2a & 0.1a & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0.1 & 0 & -0.9 & 0 & 0 \\ \hdashline 1 & 1 & 0 & 0 & 0 & 0 & 0.1 \\ 1 & 1 & 1 & -1 & 0 & -0.1 & 0 \end{array}\right) \tag{30}$$

where $a \in [0.4, 1.2]$. Let us first design a strictly proper controller if $w_2 \to z_2$ is empty in (28), which exactly corresponds to the example in [de Oliveira et al. (2002), Sec. 5.2] for $a = 1$. The design based on de Oliveira et al. (2002) with (27) comprises 32 LMI variables, while those relying on (28) with the empty channels $w_2 \to z_2$, $w_{c,2} \to z_{c,2}$ involves 36. As depicted in Fig. 2 (full blue), both designs lead to the same optimal bound γ_{opt} of the squared $\mathcal{H}_2$-norm for $a \in [0.4, 1.2]$; as expected from Section 4.1, γ_{opt} computed for (28) is less than or equal to those obtained for (27). Moreover, we can even design a strictly proper, structured $\mathcal{H}_2$-controller (28) with rational dependence on d_τ for the full matrices in (30). To the best knowledge of the authors, this cannot be handled by other techniques in this generality since finiteness of the $\mathcal{H}_2$-norm has to be guaranteed by solving a specific structured design problem. The results for our example are depicted in Fig. 2 (dashed red) and involve 88 LMI variables.

4.3 Networked $\mathcal{H}_2$-Gain-Scheduling

The recent approach in Rösinger and Scherer (2017) handles $\mathcal{H}_\infty$-problems for networks of LTI systems with delayed couplings. This is based on Fig. 3 with δ-independent transfer matrices $\tilde{G} = \tilde{G}(\delta)$, $K = K(\delta)$, while K and some parameter $F \in \mathcal{F}$ have to be designed; here $\mathcal{F}$ is a set of structured real matrices determined by the delayed coupling and admitting an LMI representation. In particular, this approach can be also applied in continuous-time with

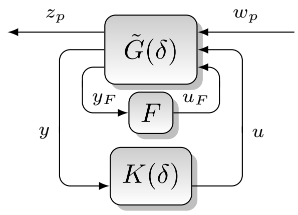

Fig. 3. Configuration for parametric $\mathcal{H}_2$-gain-scheduling.

an $\mathcal{H}_2$-cost criterion, e.g., by using the delay approximation $D_T(s) := 1/(1 + sT)$ with delay-time T which differs from the delay interpretation used in Section 4.1 and 4.2. As a step towards networked LPV systems, this motivates to consider Fig. 3 with $\tilde{G}(\delta)$ and $K(\delta)$ being scheduled by $\delta \in \mathbf{V}$. Further, we assume that $\tilde{G}(\delta)$ is structured as in

$$\begin{pmatrix} z_p \\ y \\ \hdashline y_F \end{pmatrix} = \left(\begin{array}{cc:c} \tilde{G}_{11}(\delta) & \tilde{G}_{12}(\delta) & \tilde{G}_{13}(\delta) \\ \tilde{G}_{21}(\delta) & \tilde{G}_{22}(\delta) & 0 \\ \hdashline \tilde{G}_{31}(\delta) & 0 & 0 \end{array}\right) \begin{pmatrix} w_p \\ u \\ \hdashline u_F \end{pmatrix} \tag{31}$$

with LPV systems $\tilde{G}_{ij}(\delta)$ depending on δ in the same fashion as in (1), while the particular zero structure in (31) is motivated by Rösinger and Scherer (2017) for reasons of convexification.

As in Section 2, we assume that the direct feedthrough block of $w_p \to z_p$ is zero, while that of $u \to y$ is zero for $\delta = 0$. In addition, let also those of $u_F \to z_p$ and $w_p \to y_F$ be zero. After interconnecting (31) with $u_F = F y_F$, $F \in \mathcal{F}$, we get (1) with an F-dependent operator block $G_{11}(\delta)$ whose direct feedthrough term vanishes.

Note that we can realize each $\tilde{G}_{ij}(\delta)$ with an LFR as done for $G(\delta)$ in (3); especially, suitable zero matrices appear for those blocks where we impose an assumption for the direct feedthrough term. After composing these LFRs, this allows to express the interconnection of $\tilde{G}(\delta)$ with $u_F = F y_F$ by

$$\begin{pmatrix} \dot{x} \\ \hline z_1 \\ \hline z_2 \\ \hdashline z_p \\ y \end{pmatrix} = \left(\begin{array}{ccc|ccc:cc} * & 0 & 0 & * & 0 & 0 & * & 0 \\ * & * & 0 & * & * & 0 & * & * \\ \tilde{A}_{11}(F) & * & * & \tilde{A}_{12}(F) & * & * & \tilde{B}_1^p(F) & * \\ \hline * & 0 & 0 & * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 & * \\ \hline \tilde{A}_{21}(F) & * & * & \tilde{A}_{22}(F) & * & * & \tilde{B}_2^p(F) & * \\ \hdashline \tilde{C}_1^p(F) & * & * & \tilde{C}_2^p(F) & * & 0 & 0 & D_1 \\ * & * & 0 & * & * & 0 & D_2 & 0 \end{array}\right) \begin{pmatrix} x \\ \hline w_1 \\ \hline w_2 \\ \hdashline w_p \\ u \end{pmatrix}$$

$$=: \left(\begin{array}{c|c:cc} A_{11}(F) & A_{12}(F) & B_1^p(F) & B_1 \\ \hline A_{21}(F) & A_{22}(F) & B_2^p(F) & B_2 \\ \hdashline C_1^p(F) & C_2^p(F) & D^p & D_1 \\ C_1 & C_2 & D_2 & D_3 \end{array}\right) \begin{pmatrix} x \\ \hline w_1 \\ \hline w_2 \\ \hdashline w_p \\ u \end{pmatrix}; \tag{32}$$

here $\tilde{A}_{ij}(F)$, $\tilde{B}_j^p(F)$, $\tilde{C}_i^p(F)$ depend affinely on F for $i = 1, 2$, while F-independent real matrices are displayed by $*$. We emphasize again that the structure of (32) induced by the grey lines (e.g. the triangular structure of $A_{22}(F)$) reflects the assumptions for the direct feedthrough terms. By proceeding similarly to Section 2, we work with (32) for $\delta \in \mathbf{\Delta}$ and describe the controller component $K(\delta)$ again by (5) such that the controlled interconnection for (32) is given by (6) after properly adjoining the argument F to (7). This leads to the following synthesis problem which is called *parametric $\mathcal{H}_2$-gain scheduling*.

Problem 7. For $\gamma > 0$, determine a controller $K(\delta)$ as in (5) and $F \in \mathcal{F}$ such that (G1), (G2) from Problem 2 hold.

For reasons of space, we only discuss the solution of Problem 7 for positive real scalings. Note that an application of Theorem 4 leads to the same inequalities (12)-(14) with the difference that A_{ij}, B_i^p and C_j^p in the variable blocks (15) depend affinely on the to-be-designed parameter F; in particular, if viewing F as a design variable, nonlinear expressions like $A_{ij}(F)Y_j$ appear in (15). To overcome this problem, we introduce for $i = 1,2$ the variables $R^i := (R^i_{kl})_{1\le k,l\le 3}$, $S^i := (S^i_{kl})_{1\le k,l\le 3}$ which have the same size and partition as $A_{ii}(F)$ in (32). We assume that R^i, S^i are symmetric for $i = 1$ (c.f. Scherer (2000a)). As a new condition for positive real scalings, the unknowns R^i, S^i are assumed to be PR for $i = 2$. Moreover, let us define

$$R_i^1 := \begin{pmatrix} R^i_{11} & 0 & 0 \\ R^i_{21} & I & 0 \\ R^i_{31} & 0 & I \end{pmatrix}, \qquad R_i^2 := \begin{pmatrix} I & R^i_{12} & R^i_{13} \\ 0 & R^i_{22} & R^i_{23} \\ 0 & R^i_{32} & R^i_{33} \end{pmatrix},$$
$$S_i^1 := \begin{pmatrix} S^i_{11} & S^i_{12} & S^i_{13} \\ S^i_{21} & S^i_{22} & S^i_{23} \\ 0 & 0 & I \end{pmatrix}, \quad S_i^2 := \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ S^i_{31} & S^i_{32} & S^i_{33} \end{pmatrix} \tag{33}$$

for $i = 1,2$ as well as, with (10), the blocks

$$X_2^s := \begin{pmatrix} X_{22} & S_2^1 \end{pmatrix} \text{ and } Y_2^r := \begin{pmatrix} (R_2^2)^T & Y_{22} \end{pmatrix}. \tag{34}$$

By abbreviating $X_1^s := S_1^1$ and $Y_1^r := (R_1^2)^T$, we can compactly express the following synthesis result.

Theorem 8. Let P_p and $Z \succ 0$ with $\mathrm{tr}(Z) < 1$ be given as in Theorem 3. There exists a controller (5) and $F \in \mathcal{F}$ such that the inequalities (8) are satisfied for the closed-loop matrices (7) obtained from (32) and (5), $\mathcal{X}_1 \succ 0$ as well as for $P_1 = P_2 \in \mathcal{R}$ iff there exist structured R_i^1, R_i^2, S_i^1, S_i^2 from (33), structured $\hat{X}_2$, $\hat{Y}_2$, $\hat{Z}_1$, $\hat{Z}_2$ from (10), as well as K_{ij}, L_i, M_j from (11) satisfying the inequalities

$$\mathrm{He}\big(R_2^1(R_2^2)^T\big) \succ 0, \;\; \mathrm{He}\big((S_2^2)^T S_2^1\big) \succ 0, \;\; \mathrm{He}\begin{pmatrix} \hat{X}_2 & \hat{Z}_2 \\ \hat{Z}_1 & \hat{Y}_2 \end{pmatrix} \succ 0 \tag{35}$$

and (12), (13) after inserting for $i,j = 1,2$ the blocks

$$\boldsymbol{X}_1 := \begin{pmatrix} R_1^1(R_1^2)^T & * \\ (S_1^2)^T(R_1^1)^T & (S_1^2)^T S_1^1 \end{pmatrix},$$
$$\left(\begin{array}{c:c} \boldsymbol{A}_{ij} & \boldsymbol{B}_i \\ \hdashline \boldsymbol{C}_j & \boldsymbol{D} \end{array}\right) := \left(\begin{array}{cc} 0 & B_i \\ I & 0 \\ \hdashline 0 & D_1 \end{array}\right) \begin{pmatrix} K_{ij} & L_i \\ M_j & N \end{pmatrix} \left(\begin{array}{cc:c} I & 0 & 0 \\ 0 & C_j & D_2 \end{array}\right) +$$
$$+ \left(\begin{array}{cc:c} R_i^1 A_{ij}(F) Y_j^r & R_i^1 A_{ij}(F) S_j^2 & R_i^1 B_i^p(F) \\ 0 & (X_i^s)^T A_{ij}(F) S_j^2 & (X_i^s)^T B_i^p(F) \\ \hdashline C_j^p(F) Y_j^r & C_j^p(F) S_j^2 & D^p \end{array}\right).$$

Indeed, let us stress that (35) is affine in the decision variables which leads again to a standard LMI test. For reasons of space, we only present a brief sketch of the proof of Theorem 8. The essential part is a suitable structured factorization for PR matrices from the Appendix B that extends those presented in Scherer (2000a) and allows to treat symmetric and PR matrices in the same fashion, while taking care of the F-dependent structure in (32).

Proof. The proof follows by applying suitable congruence transformations with R_i^1, R_i^2, S_i^1, S_i^2 from (33) to the inequalities in (12)-(14), and by exploiting, with Lemma 9, the factorizations $R_1^1 Y_1^T = R_1^2$, $R_2^1 Y_{21}^T = R_2^2$ and $X_1 S_1^2 = S_1^1$, $X_{23} S_2^2 = S_2^1$ along with suitable variable substitutions on (11). Note that a direct calculation shows that the congruence transformations do not influence the block $X_{22}^T A_{22} Y_{22}$ of (11) (c.f. Rösinger and Scherer (2017)). ■

5. CONCLUSION AND OUTLOOK

We have presented a parameter transformation approach for the $\mathcal{H}_2$-gain-scheduling problem with D- and positive real scalings. A key step is to solve a structured design framework to guarantee finiteness of the $\mathcal{H}_2$-norm by design. Further, the application of the setting is demonstrated for delayed systems and we also sketched an extension to handling networked LPV controller design. A future goal is the inclusion of robust $\mathcal{H}_2$-gain-scheduling problems for networked LPV systems into our framework.

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Appendix A. FACTORIZATION 1

The following factorization is part of the proof of Theorem 4 and extends the version in Scherer (2014) for symmetric matrices to the positive real case. For reasons of space, we only present the relevant modifications.

Let $\mathcal{X}_2$ being PR with $r_i^c \geq r^s$, $i = 1, 2$, and partitioned as

$$\mathcal{X}_2 = \left(\begin{array}{c|cc} X_{23} & W_{13} & W_{23} \\ \hline U_{13} & Z_{11} & Z_{12} \\ U_{23} & Z_{21} & Z_{22} \end{array}\right) \in \mathbb{R}^{(r^s+r_1^c+r_2^c)\times(r^s+r_1^c+r_2^c)}.$$

This implies that X_{23}, $\left(\begin{smallmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22}\end{smallmatrix}\right)$ and Z_{22} are PR. Hence $\left(\begin{smallmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22}\end{smallmatrix}\right)^{-1}$ and Z_{22}^{-1} are PR and, since U_{13} and U_{23} are tall matrices, we can guarantee by perturbation that

$$H_1 := \begin{pmatrix} I & 0 \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22}\end{pmatrix}^{-1}\begin{pmatrix} U_{13} \\ U_{23}\end{pmatrix} \text{ and } H_2 := Z_{22}^{-1}U_{23}$$

have full column rank such that (8) is satisfied for $\mathcal{X}_1 \succ 0$ and $\mathrm{He}(\mathcal{X}_2) \succ 0$. Let us define

$$\left(\begin{array}{c} Y_{21} \\ \hline V_{11} \\ V_{21}\end{array}\right) := \left(\begin{array}{c|cc} X_{23} & W_{13} & W_{23} \\ \hline U_{13} & Z_{11} & Z_{12} \\ U_{23} & Z_{21} & Z_{22} \end{array}\right)^{-1}\left(\begin{array}{c} I_{r^s} \\ \hline 0 \\ 0\end{array}\right)$$

$$\text{and} \quad \left(\begin{array}{c} \tilde{Y}_{22} \\ \hline \tilde{V}_{22}\end{array}\right) := \left(\begin{array}{c|c} X_{23} & W_{23} \\ \hline U_{23} & Z_{22}\end{array}\right)^{-1}\left(\begin{array}{c} I_{r^s} \\ \hline 0\end{array}\right) \tag{A.1}$$

in order to infer for some suitable $\tilde{U}_{12}$ the relation

$$\left(\begin{array}{c|cc} X_{23} & W_{13} & W_{23} \\ \hline U_{13} & Z_{11} & Z_{12} \\ U_{23} & Z_{21} & Z_{22} \end{array}\right)\left(\begin{array}{c|cc} Y_{21} & \tilde{Y}_{22} & I_{r^s} \\ \hline V_{11} & 0 & 0 \\ V_{21} & \tilde{V}_{22} & 0 \end{array}\right) = \left(\begin{array}{c|cc} I_{r^s} & I_{r^s} & X_{23} \\ \hline 0 & \tilde{U}_{12} & U_{13} \\ 0 & 0 & U_{23} \end{array}\right). \tag{A.2}$$

Since $\left(\begin{smallmatrix} X_{23} & W_{23} \\ U_{23} & Z_{22}\end{smallmatrix}\right)$ is PR, also $X_{23} - W_{23}Z_{22}^{-1}U_{23} = \tilde{Y}_{22}^{-1}$ is PR; the latter equation follows from the block-inversion formula and (A.1). Hence we infer that $\tilde{Y}_{22}$ is PR.

Let us solve $\tilde{Y}_{22}X_{22} = Y_{22}$ for X_{22}, Y_{22} carrying the structure (10). By partitioning

$$\tilde{Y}_{22} := \begin{pmatrix} \bar{Y}_{11} & \bar{Y}_{12} \\ \bar{Y}_{21} & \bar{Y}_{22}\end{pmatrix} \in \mathbb{R}^{(r_1^s+r_2^s)\times(r_1^s+r_2^s)}$$

we can indeed guarantee this relation by defining

$$\hat{X}_2 := \bar{Y}_{11}^{-1}, \qquad \hat{Y}_2 := \bar{Y}_{22} - \bar{Y}_{21}\bar{Y}_{11}^{-1}\bar{Y}_{12},$$
$$\hat{Z}_1 := \bar{Y}_{21}\bar{Y}_{11}^{-1}, \quad \hat{Z}_2 := -\bar{Y}_{11}^{-1}\bar{Y}_{12};$$

note that $\bar{Y}_{11}$ is invertible since $\tilde{Y}_{22}$ is PR. Moreover, the PR property of $\tilde{Y}_{22}$ implies that $\hat{X}_2$ and $\hat{Y}_2$ are PR by definition. By setting $V_{22} := \tilde{V}_{22}X_{22}$ and $U_{12} := \tilde{U}_{12}X_{22}$, a right-multiplication of (A.2) with X_{22} leads to

$$\mathcal{X}_2\left(\begin{array}{ccc} Y_{21} & Y_{22} & I \\ \hline V_{11} & 0 & 0 \\ V_{21} & V_{22} & 0\end{array}\right) = \left(\begin{array}{ccc} I & X_{22} & X_{23} \\ \hline 0 & U_{12} & U_{13} \\ 0 & 0 & U_{23}\end{array}\right)$$

which reads as the factorization (16) for $i = 2$ with

$$\left(\begin{array}{c} X_2 \\ \hline U_2\end{array}\right) := \left(\begin{array}{cc} X_{22} & X_{23} \\ \hline U_{12} & U_{13} \\ 0 & U_{23}\end{array}\right) \text{ and } \left(\begin{array}{c} Y_2 \\ \hline V_2\end{array}\right) := \left(\begin{array}{cc} Y_{21} & Y_{22} \\ \hline V_{11} & 0 \\ V_{21} & V_{22}\end{array}\right).$$

Further, it can be argued analogously to Scherer (2014) that $\mathcal{Y}_2$ has full column rank.

Appendix B. FACTORIZATION 2

We use a second factorization for the proof of Theorem 8.

Lemma 9. For quadratic A, Y and invertible C, X let us define the transformations

$$T_1\colon \begin{pmatrix} X & Z \\ \tilde{Z} & Y\end{pmatrix} \mapsto \begin{pmatrix} Q & S \\ \tilde{S} & R\end{pmatrix} := \begin{pmatrix} X^{-1} & X^{-1}Z \\ -\tilde{Z}X^{-1} & Y - \tilde{Z}X^{-1}Z\end{pmatrix},$$
$$T_2\colon \begin{pmatrix} A & B \\ \tilde{B} & C\end{pmatrix} \mapsto \begin{pmatrix} U & V \\ \tilde{V} & W\end{pmatrix} := \begin{pmatrix} A - BC^{-1}\tilde{B} & BC^{-1} \\ -C^{-1}\tilde{B} & C^{-1}\end{pmatrix} \tag{B.1}$$

which map the set of PR matrices (positive definite matrices) into the set of PR matrices (matrices which have positive definite diagonal blocks). Further, both maps are involutions, i.e. they are bijective with $T_i^{-1} = T_i$ for $i = 1, 2$, and describe the factorizations

$$\begin{pmatrix} Q & 0 \\ \tilde{S} & I\end{pmatrix}\begin{pmatrix} X & Z \\ \tilde{Z} & Y\end{pmatrix} = \begin{pmatrix} I & S \\ 0 & R\end{pmatrix}, \quad \begin{pmatrix} A & B \\ \tilde{B} & C\end{pmatrix}\begin{pmatrix} I & 0 \\ \tilde{V} & W\end{pmatrix} = \begin{pmatrix} U & V \\ 0 & I\end{pmatrix}. \tag{B.2}$$

Proof. The involutions property and the factorizations in (B.2) follow by a direct calculation. Let us show that T_1 preserves the PR property. Now $\left(\begin{smallmatrix} X & Z \\ \tilde{Z} & Y\end{smallmatrix}\right)$ is PR iff

$$\begin{pmatrix} -X^{-1} & -X^{-1}Z \\ 0 & I\end{pmatrix}^T\begin{pmatrix} X & Z \\ \tilde{Z} & Y\end{pmatrix}\begin{pmatrix} -X^{-1} & -X^{-1}Z \\ 0 & I\end{pmatrix} =$$
$$= \begin{pmatrix} X^{-T} & 0 \\ (X^{-1}Z)^T - \tilde{Z}X^{-1} & Y - \tilde{Z}X^{-1}Z\end{pmatrix} \text{ is PR.}$$

The latter is satisfied iff

$$\mathrm{He}\begin{pmatrix} X^{-T} & 0 \\ (X^{-1}Z)^T - \tilde{Z}X^{-1} & Y - \tilde{Z}X^{-1}Z\end{pmatrix} =$$
$$= \mathrm{He}\begin{pmatrix} X^{-1} & X^{-1}Z \\ -\tilde{Z}X^{-1} & Y - \tilde{Z}X^{-1}Z\end{pmatrix} \succ 0$$

which is equivalent to $\left(\begin{smallmatrix} X^{-1} & X^{-1}Z \\ -\tilde{Z}X^{-1} & Y-\tilde{Z}X^{-1}Z\end{smallmatrix}\right)$ being PR. By similar arguments, T_2 preserves the PR property, too. Moreover, standard Schur complement arguments show that T_i bijectively map positive definite matrices to the set of matrices with positive definite diagonal blocks. Therefore, the maps T_i in (B.1) are well-defined. ■