Theory of Robust Control

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Contents

1 Introduction to Basic Concepts

1.1 Systems and Signals

In these notes we intend to develop the theory of robust control for linear time invariant finite-dimensional systems that are briefly called LTI-systems. Recall that such systems are described in the state-space as

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{aligned} \tag{1.1}$$

with input u, output y, state x, and real matrices A, B, C, D of suitable size.

Here u, y, x are signals. Signals are functions of time $t \in [0, \infty)$ that are piece-wise continuous. (On finite intervals, such signals have only finitely many jumps as discontinuities.) They can either take their values in $\mathbb{R}$, or they can have k components such that they take their values in $\mathbb{R}^k$. To clearly identify e.g. x as a signal, we sometimes write $x(.)$ to stress this point. (Recall that $x(.)$ denotes the signal as a whole, whereas $x(t)$ denotes the value of the signal at the time-instant $t \geq 0$.)

Somewhat more precisely, (1.1) is defined by $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times n}$. Moreover, for an input trajectory $u : [0, \infty) \to \mathbb{R}^m$, there exists a unique state response $x : [0, \infty) \to \mathbb{R}^n$ of the system (1.1) which has the output $y : [0, \infty) \to \mathbb{R}^k$. It is well-known that the output response $y(.)$ to the input $u(.)$ is given by

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t) \text{ for } t \geq 0.$$

We do not repeat the standard notions of controllability of the system or of the pair (A, B), and of observability of the system or of (A, C). Nevertheless we recall the following very basic facts:

- The **Hautus test for controllability**: (A, B) is controllable if and only if the matrix

$$\begin{pmatrix} A - \lambda I & B \end{pmatrix}$$

  has full row rank for all $\lambda \in \mathbb{C}$.

- The **Hautus test for observability**: (A, C) is observable if and only if the matrix

$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

  has full column rank for all $\lambda \in \mathbb{C}$.

- The system (1.1) or (A, B) is said to be **stabilizable** if there exists a feedback matrix F such that $A + BF$ has all its eigenvalues in the open left-half plane $\mathbb{C}_< := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$.

  Recall the **Hautus test for stabilizability**: (A, B) is stabilizable if and only if the matrix
$$\begin{pmatrix} A - \lambda I & B \end{pmatrix}$$
  has full column rank for all $\lambda \in \mathbb{C}_\geq$, where $\mathbb{C}_\geq$ denotes the closed right half-plane $\mathbb{C}_\geq := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$.

- The system (1.1) or (A, C) is said to be **detectable** if there exists an L such that $A + LC$ has all its eigenvalues in the open left-half plane $\mathbb{C}_<$.

  Recall the **Hautus test for detectability**: (A, C) is detectable if and only if the matrix
$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$
  has full column rank for all $\lambda \in \mathbb{C}_\geq$

The **transfer matrix** $G(s)$ of the system (1.1) is defined as

$$G(s) = C(sI - A)^{-1}B + D$$

and is a matrix whose elements consist of real-rational and proper functions in s. Any such function is a fraction of two polynomials in s with real coefficients; properness of such a function means that the degree of the numerator is not larger than the degree of the denominator.

Why does the transfer matrix pop up? Suppose the input signal $u(.)$ has the Laplace-transform
$$\hat{u}(s) = \int_0^\infty e^{-st} u(t)\, dt.$$
Then the output $y(.)$ of (1.1) also has a Laplace transform that can be calculated as

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]\hat{u}(s).$$

For $x_0 = 0$ (which means that the state of the system starts with value zero at time 0), the relation between the Laplace transform of the input and the output signals is hence given by the transfer matrix as follows:

$$\hat{y}(s) = G(s)\hat{u}(s).$$

The 'complicated' convolution integral in the time-domain is transformed into a the 'simpler' multiplication operation in the frequency domain.

We have briefly addressed two different ways of representing a system: One representation in the state-space defined with specific constant matrices A, B, C, D, and one in the frequency domain defined via a real-rational proper transfer matrix $G(s)$.

Remark 1.1 In this course we want to view a system as a device that processes signals; hence a system is nothing but a mapping that maps the input signal $u(.)$ into the output signal $y(.)$ (for a certain initial condition). One should distinguish the system (the mapping) from its representations, such as the one in the state-space via A, B, C, D, or that in the frequency domain via $G(s)$. System properties should be formulated in terms of how signals are processed, and system representations are used to formulate algorithms how certain system properties can be verified.

The fundamental relation between the state-space and frequency domain representation is investigated in the so-called realization theory. Moving from the state-space to the frequency domain just requires to calculate the transfer matrix $G(s)$.

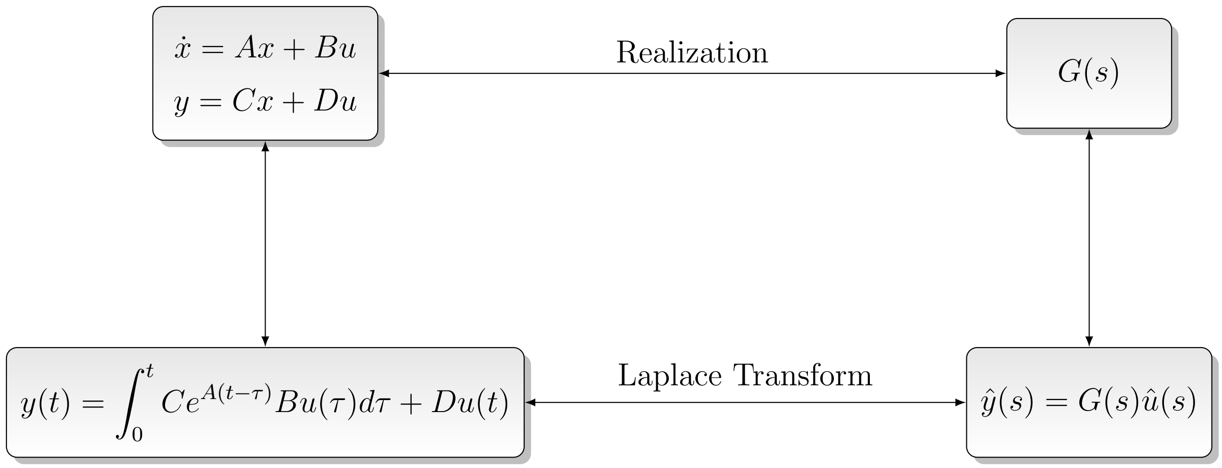
Conversely, suppose $H(s)$ is an arbitrary matrix whose elements are real-rational proper functions. Then there **always** exist real matrices A_H, B_H, C_H, D_H such that

$$H(s) = C_H(sI - A_H)^{-1}B_H + D_H$$

holds true. This representation of the transfer matrix is called a **realization**. Realizations are not unique. Even more importantly, the size of the matrix A_H can vary for various realizations. However, there are realizations where A_H is of minimal size, the so-called minimal realization. There is a simple answer to the question of whether a realization is minimal: This happens if and only if (A_H, B_H) is controllable and (A_H, C_H) is observable.

Task. Recapitulate how you can compute a minimal realization of an arbitrary real rational proper transfer matrix H.

Pictorially, this discussion about the system representations in the time- and frequency-domain and the interpretation as a mapping of signals (for a zero initial condition of the state) can be depicted as follows:



We use the symbol

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

both for the system mapping $u \to y$ as defined via the differential equation with initial condition 0, and for the corresponding transfer matrix G.

1.2 Stability of LTI Systems

Recall that any matrix $H(s)$ whose elements are real rational functions is **stable** if

- $H(s)$ is proper (there is no pole at infinity) and
- $H(s)$ has only poles in the open left-half plane $\mathbb{C}_<$ (i.e. there is no pole in the closed right half plane $\mathbb{C}_\geq$).

Properness means that, for all entries of H, the degree of the numerator polynomial is not larger than the degree of the denominator polynomial. H is called *strictly proper* if the degree of the numerator polynomial is strictly smaller than the degree of the denominator polynomial for all elements of H. Note that strict properness is equivalent to $\lim_{|s|\to\infty} H(s) = 0$.

For the set of real rational proper and stable matrices of dimension $k \times m$ we use the special symbol

$$RH_\infty^{k\times m}$$

and if the dimension is understood from the context we simply write RH_∞. Recall that the three most important operations performed on stable transfer matrices do not lead us out of this set: A scalar multiple of one stable transfer matrix as well as the sum and the product of two stable transfer matrices (of compatible dimensions) are stable.

On the other hand, the state-space system (1.1) is said to be **stable** if A has all its eigenvalues in the open left-half plane $\mathbb{C}_<$. We will denote the set of eigenvalues of A by $\mathrm{eig}(A)$, the spectrum of A. Then stability of (1.1) is simply expressed as

$$\mathrm{eig}(A) \subset \mathbb{C}_<.$$

We say as well that the *matrix* A is stable (or Hurwitz) if it has this property.

We recall the following relation between the stability of the system (1.1) and the stability of the corresponding transfer matrix $G(s) = C(sI - A)^{-1}B + D$:

- If (1.1) (or A) is stable, then $G(s)$ is stable.
- Conversely, if $G(s)$ is stable, if (A, B) is stabilizable, and if (A, C) is detectable, then (1.1) (or A) is stable.

Note that all these definitions are given in terms of properties of the representation. Nevertheless, these concepts are closely related - at least for LTI systems - to the so-called bounded-input bounded-output stability properties.

A vector valued signal $u(.)$ is bounded if the **maximal amplitude** or **peak**

$$\|u\|_\infty = \sup_{t \geq 0} \|u(t)\| \quad \text{is finite.}$$

Here $\|u(t)\|$ just equals the Euclidean norm $\sqrt{u(t)^T u(t)}$ of the vector $u(t)$. The symbol $\|u\|_\infty$ for the peak indicates that the peak is, in fact, a norm on the vector space of all bounded signals; it is called the L_∞-norm.

The system (1.1) is said to be bounded-input bounded-output (BIBO) stable if it maps an arbitrary bounded input $u(.)$ into an output that is bounded as well. In short, $\|u\|_\infty < \infty$ implies $\|y\|_\infty < \infty$. It is an interesting fact that, for LTI systems, BIBO stability is equivalent to the stability of the corresponding transfer matrix as defined earlier.

Theorem 1.2 *The system (1.1) maps bounded inputs $u(.)$ into bounded outputs $y(.)$ if and only if the corresponding transfer matrix $C(sI - A)^{-1}B + D$ is stable.*

To summarize, for a stabilizable and detectable realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ of an LTI system, the following notions are equivalent: Stability of the system (1.1), stability of the corresponding transfer matrix $C(sI - A)^{-1}B + D$, and BIBO stability of the system (1.1) viewed as an input output mapping.

Stability is a qualitative property. Another important issue is to quantify in how far signals are amplified or quenched by a system. If we look at one input $u(.) \neq 0$, and if we take $\|u\|_\infty$ and $\|y\|_\infty$ as a measure of size for the input and the output of the system (1.1), the amplification for this specific input signal is nothing but

$$\frac{\|y\|_\infty}{\|u\|_\infty}.$$

The worst possible amplification is obtained by finding the largest of these quotients if varying $u(.)$ over **all** bounded signals:

$$\gamma_{\text{peak}} = \sup_{0<\|u\|_\infty<\infty} \frac{\|y\|_\infty}{\|u\|_\infty}. \tag{1.2}$$

This is the so-called **peak-to-peak gain** of the system (1.1). Then it just follows from the definition that

$$\|y\|_\infty \le \gamma_{\text{peak}} \|u\|_\infty$$

holds for all bounded input signals $u(.)$: Hence γ_{peak} quantifies how the amplitudes of the bounded input signals are amplified or quenched by the system. Since γ_{peak} is, in fact, the smallest number such that this inequality is satisfied, there does exist an input signal such that the peak amplification is actually arbitrarily close to γ_{peak}. (The supremum in (1.2) is not necessarily attained by some input signal. Hence we cannot say that γ_{peak} is attained, but we can come arbitrarily close.)

Besides the peak, we could as well work with the energy of a signal $x(.)$, defined as

$$\|x\|_2 = \sqrt{\int_0^\infty \|x(t)\|^2 \, dt},$$

to measure its size. Note that a signal with a large energy can have a small peak and vice versa. (Think of examples!) Hence we are really talking about different physical motivations if deciding for $\|.\|_\infty$ or for $\|.\|_2$ as a measure of size.

Now the question arises when a system maps any signal of finite energy again into a signal of finite energy; in short:

$$\|u\|_2 < \infty \quad \text{implies} \quad \|y\|_2 < \infty.$$

It is somewhat surprising that, for the system (1.1), this property is again equivalent to the stability of the corresponding transfer matrix $C(sI-A)^{-1}B+D$. Hence the qualitative property of BIBO stability does not depend on whether one chooses the peak $\|.\|_\infty$ or the energy $\|.\|_2$ to characterize boundedness of a signal.

Remark 1.3 Note that this is a fundamental property of LTI system that is by no means valid for other type of systems, even if they admit a state-space realization such as non-linear system defined via differential equations.

Although the qualitative property of stability does not depend on the chosen measure of size for the signals, the quantitative measure for the system amplification, the system gain, is highly dependent on the chosen norm. The **energy gain** of (1.1) is analogously defined as for the peak-to-peak gain defined by

$$\gamma_{\text{energy}} = \sup_{0<\|u\|_2<\infty} \frac{\|y\|_2}{\|u\|_2}.$$

Contrary to the peak-to-peak gain, one can nicely relate the energy gain of the system (1.1) to the transfer matrix of the system. In fact, one can prove that γ_{energy} is equal to the largest value of

$$\sigma_{\max}(G(i\omega)) = \|G(i\omega)\|$$

f varying the frequency as $\omega \in \mathbb{R}$. Here $\sigma_{\max}(A)$ denotes the maximum singular value of the matrix A which equals the spectral norm of A. Let us hence introduce the abbreviation

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(i\omega)) = \sup_{\omega\in\mathbb{R}} \|G(i\omega)\|.$$

As indicated by the symbol, this formula defines a norm on the vector space of all real-rational proper and stable matrices $RH_\infty^{k\times m}$ which is called the H_∞-norm.

We can conclude that the energy gain of the stable LTI system (1.1) is just equal to the H_∞-norm of the corresponding transfer matrix:

$$\gamma_{\text{energy}} = \|G\|_\infty.$$

1.3 Stable Inverses

For any real-rational *matrix* $G(s)$, we can compute the real rational *function* $\det(G(s))$. It is well-known that $G(s)$ has a real-rational inverse if and only if $\det(G(s))$ is not the zero function (does not vanish identically). If $G(s)$ is proper, it is easy to verify that it has a *proper inverse* if and only if $\det(G(\infty))$ (which is well-defined since $G(\infty)$ is just a real matrix) does not vanish.

The goal is to derive a similar condition for the proper and stable $G(s)$ to have a proper and stable inverse. Here is the desired characterization.

Lemma 1.4 *The proper and stable matrix $G(s)$ has a proper and stable inverse if and only if the matrix $G(\infty)$ is non-singular, and the rational function $\det(G(s))$ does not have any zeros in the closed right-half plane.*

Proof. Assume that $G(s)$ has the proper and stable inverse $H(s)$. From $G(s)H(s) = I$ we infer $\det(G(s))\det(H(s)) = 1$ or

$$\det(G(s)) = \frac{1}{\det(H(s))}.$$

Since $H(s)$ is stable, $\det(H(s))$ has no poles in $\mathbb{C}_{\geq} \cup \{\infty\}$. Therefore, the reciprocal rational function and hence $\det(G(s))$ does not have zeros in this set.

Conversely, let us assume that $\det(G(s))$ has no zeros in $\mathbb{C}_{\geq} \cup \{\infty\}$. Then clearly

$$\frac{1}{\det(G(s))} \text{ is proper and stable.}$$

Now recall that the inverse of $G(s)$ is given by the formula

$$G(s)^{-1} = \frac{1}{\det(G(s))}\operatorname{adj}(G(s))$$

where $\operatorname{adj}(G(s))$ denotes the algebraic adjoint of $G(s)$. These adjoints are computed by taking products and sums/differences of the elements of $G(s)$; since $G(s)$ is stable, the adjoint of $G(s)$ is, therefore, a stable matrix. Then the explicit formula for $G(s)^{-1}$ reveals that this inverse must actually be stable as well.

$\blacksquare$

Remark 1.5 It is important to apply this result to stable $G(s)$ only. For example, the proper unstable matrix

$$G(s) = \begin{pmatrix} \frac{s+1}{s+2} & \frac{1}{s-1} \\ 0 & \frac{s+2}{s+1} \end{pmatrix}$$

satisfies $\det(G(s)) = 1$ for all s. Hence, its determinant has no zeros in the closed right-half plane and at infinity; nevertheless, its inverse is not stable!

Let us now assume that the proper G has a realization

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Recall that G has a proper inverse iff $D = G(\infty)$ is invertible. If D has an inverse, the proper inverse of G admits the realization

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right].$$

Why? A signal based arguments leads directly to the answer:

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw \tag{1.3}$$

is equivalent to

$$\dot{x} = Ax + Bw, \quad w = -D^{-1}Cx + D^{-1}z$$

and hence to

$$\dot{x} = (A - BD^{-1}C)x + BD^{-1}z, \quad w = -D^{-1}Cx + D^{-1}z. \tag{1.4}$$

This gives a test of whether the stable G has a proper and stable inverse directly in terms of the matrices A, B, C, D of some realization.

Lemma 1.6 *Let $G(s)$ be stable and let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stabilizable and detectable realization. Then $G(s)$ has a proper and stable inverse if and only if D is non-singular and $A - BD^{-1}C$ is stable.*

Proof. Suppose G has the proper and stable inverse H. Then $G(\infty) = D$ is non-singular. We can hence define the system (1.4); since the realization (1.3) is stabilizable and detectable, one can easily verify (with the Hautus test) that the same is true for the realization (1.4). We have argued above that (1.4) is a realization of H; since H is stable, we can conclude that $A - BD^{-1}C$ must be stable.

The converse is easier to see: If D is non-singular and $A - BD^{-1}C$ is stable, (1.4) defines the stable transfer matrix H. Since H is the inverse of G, as was seen above, we conclude that G admits a proper and stable inverse.

■

Exercises

1) Suppose that $G(s)$ is a real-rational proper matrix. Explain in general how you can compute a state-space realization of $G(s)$ with the command `sysic` of Matlab's Robust Control Toolbox, and discuss how to obtain a minimal realization.

   (Matlab) Compute in this way a realization of

$$G(s) = \begin{pmatrix} 1/s & 1/(s+1)^2 & s/(s+1) \\ (s^2 - 3s + 5)/(2s^3 + 4s + 1) & 1/s & 1/(s+1) \end{pmatrix}.$$

2) Suppose that $u(.)$ has finite energy, let $y(.)$ be the output of (1.1), and denote the Laplace transforms of both signals by $\hat{u}(.)$ and $\hat{y}(.)$ respectively. Show that

$$\int_{-\infty}^{\infty} \hat{y}(i\omega)^* \hat{y}(i\omega)\, d\omega \leq \|G\|_\infty^2 \int_{-\infty}^{\infty} \hat{u}(i\omega)^* \hat{u}(i\omega)\, d\omega.$$

Argue that this implies $\|y\|_2 \leq \|G\|_\infty \|u\|_2$, and that this reveals that the energy gain γ_{energy} is not larger than $\|G\|_\infty$.

Can you find a sequence $u_j(.)$ of signals with finite energy such that

$$\lim_{j\to\infty} \frac{\|Gu_j\|_2}{\|u_j\|_2} = \|G\|_\infty?$$

3) Look at the system $\dot{x} = ax + u,\ y = x$. Compute the peak-to-gain of this system. Determine a worst input u, i.e., an input for which the peak of the corresponding output equals the peak-to-peak gain of the system.

4) For any discrete-time real-valued signal $x = (x_0, x_1, \ldots)$ let us define the peak as

$$\|x\|_\infty := \sup_{k\geq 0} |x_k|.$$

Consider the SISO discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k, \quad x_0 = 0, \quad k \geq 0$$

where all eigenvalues of A have absolute value smaller than 1 (discrete-time stability). As in continuous-time, the peak-to-peak gain of this system is defined as

$$\sup_{0<\|u\|_\infty<\infty} \frac{\|y\|_\infty}{\|u\|_\infty}.$$

Derive a formula for the the peak-to-peak gain of the system!

Hint: If setting $u^m := (u_0 \ \cdots \ u_m)^T$, $y^m := (y_0 \ \cdots \ y_m)^T$, determine a matrix M_m such that $y^m = M_m u^m$. How are the peak-to-peak gain of the system and the norm of M_m induced by the vector norm $\|.\|_\infty$ related?

5) Prove Theorem 1.2.

2 Robustness for SISO Systems

In this chapter robustness for single-input single-output (SISO) linear time invariant systems is considered. For this purpose we introduce, in Section 2.1, a standard tracking configuration and characterize stability of the sensitivity and complementary sensitivity transfer function.

We recall a version of the classical Nyquist criterion, which is a graphical test for checking closed-loop stability by considering the Nyquist plot of the open-loop transfer function.

Furthermore we provide a mathematically precise definition of a multiplicative uncertainty model and develop the related tests for checking robust stability and robust performance.

Finally, in Section 2.6, we address the notion of internal stabilization for a standard feedback loop that consists of SISO components only.

To start, it is convenient to remind us about some facts about coprime polynomials.

Definition 2.1 *Let p and q be real polynomials. A* **greatest common divisor** *of p and q is a monic polynomial $d = \gcd(p, q)$ that divides p and q, such that every common divisor of p and q also divides d. Two polynomials p and q are* **coprime** *if $\gcd(p, q) = 1$.*

We mainly use coprime polynomials because of property c) of the following theorem.

Theorem 2.2 *For two real polynomials p and q, the following statements are equivalent.*

a) p and q are coprime.

b) There exist real polynomials a and b such that $bp + aq = 1$.

c) p and q do not have common zeros.

Proof. b) follows from a) from the well-known lemma of Bézout.

b) $\Rightarrow$ c): If $q(s) = 0$ for $s \in \mathbb{C}$ then $b(s)p(s) = b(s)q(s) + a(s)q(s) = 1$. This implies $p(s) \neq 0$.

c) $\Rightarrow$ a): Assume that p and q are not coprime. Then there exists a polynomial d with $\deg(d) > 0$ and $\gcd(p, q) = d$. By the fundamental theorem of algebra there exists some $s \in \mathbb{C}$ with $d(s) = 0$. Since d divides p and q, s is also a common zero of p and q.

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Any real rational function G, and in particular any transfer function, can be expressed as

$$G = \frac{N}{D} \text{ with real coprime polynomials } N \text{ and } D.$$

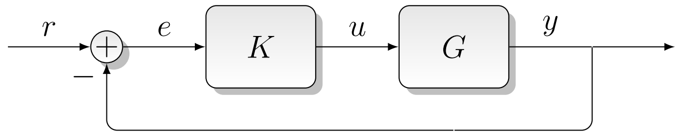

Figure 1: The standard tracking configuration.

Since N and D have no common zeros, the zeros of G in $\mathbb{C}$ are the zeros of N and the poles of G in $\mathbb{C}$ are the zeros of D, counted with the corresponding multiplicities. If not emphasized explicitly, statements about poles or zeros should always be interpreted as "counted with multiplicities". Therefore $\frac{1}{s-1}$ and $\frac{1}{(s-1)^2}$ do not have identical poles.

If the **relative degree** $r(G) := \deg(D) - \deg(N)$ of G is positive, then G is said to have a zero at ∞ with multiplicity $r(G)$. If G is not proper, which means $\deg(N) > \deg(D)$, then G is said to have a pole at ∞ of multiplicity $-r(G) = \deg(N) - \deg(D)$.

2.1 The Standard Tracking Configuration

In this section, if nothing else is mentioned, we consider the feedback interconnection

$$y = Gu, \quad u = K(r - y)$$

with the open-loop system G and the controller K. We assume that both G and K are SISO transfer functions. This interconnection is the standard tracking configuration and is depicted in Figure 1.

Roughly speaking, the standard goals in designing a controller K for the interconnection can be expressed as follows:

- Stabilize the interconnection.
- The system output y should track r well, which means that the norm of the tracking error $e = r - y$ is small.
- The control action u should not be too large.

To analyze stability, let us define the **loop transfer function** or loop-gain

$$L := GK.$$

Then the interconnection equations can be written as

$$y = Le, \quad e = r - y.$$

If the transfer function L is not identically equal to -1 we get

$$y = Tr \text{ and } e = Sr \text{ with } T := \frac{L}{1+L} \text{ and } S := \frac{1}{1+L}.$$

S is called the **sensitivity** (transfer function) and T the **complementary sensitivity** (transfer function) of the feedback loop. Note that $S+T=1$ always holds. It is also easy to see that the poles of S and T are the zeros of $1+L$.

Theorem 2.3 *The poles of S and the poles of T in $\mathbb{C}\cup\{\infty\}$ are exactly given by the zeros of $1+L$ in $\mathbb{C}\cup\{\infty\}$.*

Proof. L can be expressed as $L=\frac{N}{D}$ with real coprime polynomials N and D. Then

$$1+L=1+\frac{N}{D}=\frac{D+N}{D}$$

and thus

$$S=\frac{D}{D+N} \quad \text{as well as} \quad T=\frac{N}{D+N}.$$

If $D(\lambda)=0$, coprimeness of N and D implies $N(\lambda)\neq 0$ and therefore $D(\lambda)+N(\lambda)\neq 0$. Hence D and $D+N$ are coprime. This means that the poles of S in $\mathbb{C}$ are given by the zeros of $D+N$ in $\mathbb{C}$ which are the zeros of $1+L$ in $\mathbb{C}$.

Now let $D(s)=d_ks^k+\cdots+d_0$ and $N(s)=n_ms^m+\cdots+n_0$ with $d_k,n_m\neq 0$. Then S has a pole at ∞ iff $\deg(D)>\deg(D+N)$ iff $k=m$ and $d_k+n_m=0$ iff $L(\infty)=-1$ iff $L+1$ has a zero at ∞.

The proof for T proceeds along the same arguments.

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By Theorem 2.3, S is proper iff $1+L(\infty)\neq 0$; similarly (and since stability includes properness as a requirement), S is stable iff $1+L(\lambda)\neq 0$ for all $\lambda\in\mathbb{C}^0\cup\mathbb{C}^+\cup\{\infty\}$.

2.2 The Nyquist Stability Criterion

We are interested in stability of $(1+L)^{-1}$ if L is a given loop transfer function; in this section we proivde a classical graphical test for this property, the so-called Nyquist stability criterion. In its essence, it is a graphical test for counting the number of zeros of $1+L$ that are encircled by a Nyquist contour of L.

For this purpose we consider the grey oriented curve Γ as depicted in Figure 2 (with small semicircles of radius $r>0$ avoiding poles of L on the imaginary axis and one large semicircle of radius $R>0$ in order to capture all poles of L in $\mathbb{C}_=\cup\mathbb{C}_>$); Let Ω denote the open set that is encircled by Γ once in clockwise direction.

Definition 2.4 *The curve Γ is called a* **Nyquist contour** *for the transfer function L if it does not pass through any pole of L and if the set of poles of L in Ω is equal to the set of poles of L in $\mathbb{C}^0\cup\mathbb{C}^+$.*

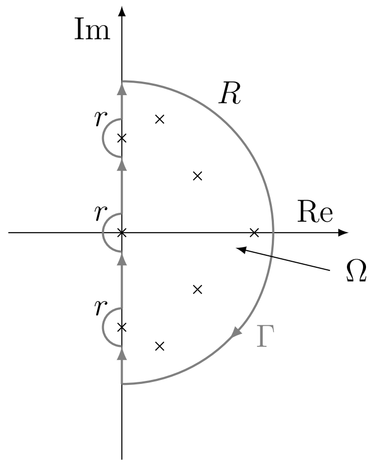


Figure 2: Nyquist contour Γ and encircled region Ω; crosses indicate poles of L in $\mathbb{C}_= \cup \mathbb{C}_>$.

If Γ does not pass through any pole of L, the remaining properties for Γ to qualify as a Nyquist contour can also be expressed as follows:

- **All** poles of L in $\mathbb{C}_= \cup \mathbb{C}_>$ are encircled by Γ exactly once in clockwise direction. (R is so large to ensure that no poles of L in $\mathbb{C}_= \cup \mathbb{C}_>$ are located outside $\Gamma \cup \Omega$).
- **No** pole of L in $\mathbb{C}_<$ is encircled by Γ. (r is sufficiently small such that all poles of L in $\mathbb{C}_<$ are located outside $\Gamma \cup \Omega$.)

Since a transfer function L has only finitely many poles in $\mathbb{C}$, we can always choose $r > 0$ small enough and $R > 0$ large enough in order to make sure that Γ is a Nyquist contour for L. Then increasing $R > 0$ and/or decreasing $r > 0$ does not change this property.

The **Nyquist plot** of L is just defined as the image of a chosen Nyquist contour Γ under L, with the orientation inherited from Γ:

$$L(\Gamma) = \{L(\lambda) : \ \lambda \in \Gamma\}.$$

Theorem 2.5 (Nyquist stability criterion). *Let L be a proper transfer function and Γ a Nyquist contour for L such that the Nyquist plot of L does not pass through -1 and $L(\lambda) \neq -1$ for $\lambda = \infty$ as well as for all $\lambda \in \mathbb{C}_= \cup \mathbb{C}_>$ with $|\lambda| > R$. Suppose that L has n_{0+} poles on the imaginary axis or in the open right-half plane (counted with multiplicities). Then $(1 + L)^{-1}$ is stable iff the Nyquist plot of L encircles -1 exactly n_{0+} times in the counterclockwise direction.*

Remark. We will see that the Nyquist plot of L always encircles -1 **at most** n_{0+} times in the counterclockwise direction, irrespective of whether $(1+L)^{-1}$ is stable or not. Stability of $(1+L)^{-1}$ is thus equivalent to the fact that the Nyquist plot of L encircles -1 **at least** n_{0+} times in the counterclockwise direction.

Proof. As abbreviations let us denote by $Z_A(g)$, $P_A(g)$ the number of zeros, poles of the function g in the set $A \subset \mathbb{C}$. We prepare the proof with the following observations involving the function $f := L + 1$:

- L and f have identical poles in $\mathbb{C}$. Hence $P_\Omega(L) = P_\Omega(f)$.
- The curve $f(\Gamma)$ is obtained from the Nyquist plot $L(\Gamma)$ through a shift by 1 (to the right) in the complex plane. Hence $f(\Gamma)$ does not pass through zero, i.e., $Z_\Gamma(f) = 0$.
- By the assumption on L, $f(\lambda) \neq 0$ for $\lambda = \infty$ and $\lambda \in \{\lambda \in \mathbb{C}_= \cup \mathbb{C}_> : \ |\lambda| \geq R\}$; the latter set is the complement of Ω in $\mathbb{C}_= \cup \mathbb{C}_>$. Therefore, $(1+L)^{-1}$ is stable iff
$$Z_\Omega(f) = 0. \tag{2.1}$$
- Since Γ is a Nyquist contour for L we have $P_\Omega(L) = P_{\mathbb{C}^0 \cup \mathbb{C}^+}(L) = n_{0+}$. Due to the first bullet this implies $P_\Omega(f) = n_{0+}$.

For the Nyquist plot of L, let N now denote the number of encirclements of -1 in the counterclockwise direction. Then $f(\Gamma)$ encircles 0 exactly $-N$ times in clockwise direction. By the principle of the argument we infer that $-N = Z_\Omega(f) - P_\Omega(f)$ and thus
$$N = n_{0+} - Z_\Omega(f). \tag{2.2}$$
(This implies that $N \leq n_{0+}$ always holds as emphasized in the remark after the theorem.)

Proof of "if". By hypothesis $N = n_{0+}$. Hence $Z_\Omega(f) = 0$ and thus $(1+L)^{-1}$ is stable. (This already follows from $N \geq n_{0+}$ which proves the statement in the remark.)

Proof of "only if". If $(1+L)^{-1}$ is stable we have (2.1). Then (2.2) shows $N = n_{0+}$.

■

Remarks.

- The result can also be applied if L is an irrational meromorphic function (i.e. the quotient of two functions that are analytic on $\mathbb{C}$). As a typically example, it can be used for loop transfer functions $L(s) = e^{-sT}H(s)$ with $T > 0$ and a transfer function H, as emerging in feedback loops with a delay.
- For transfer functions L one could set $R = \infty$: The Nyquist contour then just consists of the whole imaginary axis (still with the small semicircles to avoid poles of L on the axis and leaving all poles of L in $\mathbb{C}_<$ to the left); the Nyquist plot of L then just consists of the image under L in union with $L(\infty)$; moreover, in the first hypotheses of Theorem 2.5, no $\lambda \in \mathbb{C}_= \cup \mathbb{C}_>$ with $|\lambda| > R$ exist and it only remains to check $L(\infty) \neq -1$.

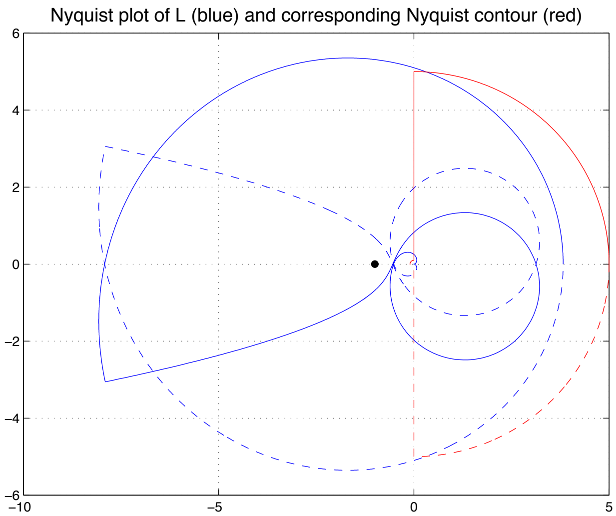


Figure 3: Nyquist plot of L for the satellite example. For the Nyquist contour we choose $r = 0.1$ and $R = 5$.

Example 2.6 The transfer function of a flexible satellite is given by

$$G(s) = \frac{0.036(s + 25.28)}{s^2(s^2 + 0.0396s + 1)}.$$

A controller which renders $S = (1 + GK)^{-1}$ stable is given by

$$K(s) = \frac{7.9212(s + 0.1818)(s^2 - 0.2244s + 0.8981)}{(s^2 + 3.899s + 4.745)(s^2 + 1.039s + 3.395)}.$$

Recall the standard construction of stabilizing controllers based on the separation principle. One takes a minimal realization (A, B, C, D) of G and chooses F and J such that $A + BF$ and $A + JC$ are Hurwitz. Then $(A + BF + JC + JDF, J, F, 0)$ is a realization of a controller K which renders $(1 + GK)^{-1}$ stable. Also recall that F and J can, for example, be computed by solving an LQR and a Kalman filter problem.

In our case the loop gain $L(s)$ equals

$$\frac{0.28516(s + 25.28)(s + 0.1818)(s^2 - 0.2244s + 0.8981)}{s^2(s^2 + 3.899s + 4.745)(s^2 + 0.0396s + 1.001)(s^2 + 1.039s + 3.395)}.$$

L is strictly proper, i.e., $L(\infty) = 0 \neq -1$. Therefore $(1 + L)^{-1}$ is proper. Moreover, L has $s = 0$ with multiplicity two as the only pole in the closed right half-plane.

The Nyquist plot for $R = 5$ and $r = 0.1$ of L in Figure 3 encircles -1 two times in the counterclockwise direction. The dashed lines correspond to negative frequencies. Hence Theorem 2.5 implies that $(1 + L)^{-1}$ has all its poles in $\mathbb{C}_<$.

Indeed, $(1+L(s))^{-1}$ equals

$$\frac{s^2(s^2+3.899s+4.745)(s^2+0.0396s+1.001)(s^2+1.039s+3.395)}{(s^2+1.322s+0.606)(s^2+1.77s+0.9899)(s^2+1.217s+1.509)(s^2+0.6677s+1.3)}.$$

2.3 Classical Robustness Indicators

We now make a first step towards considering robustness of feedback systems. Suppose that $(1+L)^{-1}$ is stable. If the loop transfer function changes from L to kL for $k \in \mathbb{R}$, we can ask ourselves whether the uncertain system $(1+kL)^{-1}$ remains stable for all uncertainties $k \in (k_-, k_+)$ for constants $0 \leq k_- < 1 < k_+ \leq \infty$. Classically, the largest $1 < k_+ \leq \infty$ and the smallest $0 \leq k_- < 1$ such that $(1+kL)^{-1}$ is stable for all $k \in (k_-, k_+)$ are called the **upper** and **lower gain-margins**, respectively.

Similar the largest $0 < \phi \leq \pi$ such that $(1+cL)^{-1}$ is stable for all c with $|c| = 1$ and $\arg(c) \in (-\phi, \phi)$ is called the **phase-margin**.

By the Nyquist stability criterion and corresponding remarks, these margins can be read off from those points where the Nyquist plot of L crosses the negative real axis/the unit circle. If changing L into kL for real $k > 0$ or into cL for complex $|c| = 1$, the effect onto the Nyquist plot can be easily derived from the well-known relation

$$|cL(i\omega)| = |c||L(i\omega)| \quad \text{and} \quad \arg(cL(i\omega)) = \arg(c) + \arg(L(i\omega)) \quad \text{for any} \quad c \in \mathbb{C},$$

with the usual interpretation of the equation on the right. Roughly speaking, if continuously deforming L into cL, stability will be lost at that value of c for which the Nyquist plot of cL happens to pass through -1 at some frequency $\omega \in \mathbb{R}$, since then $(1+cL)^{-1}$ has a pole at $i\omega$. For the gain-margins, we thus have to consider those frequencies for which the Nyquist plot of L crosses the negative real axis, while for the phase-margin the crossing points with the unit circle are relevant, as illustrated in in Figure 4.

Example 2.7 Recall the transfer functions from Example 2.6 and the Nyquist plot in Figure 3, a zoomed-in version of which is given in Figure 5. The relevant crossing point on the real axis is $x_1 \approx -0.54$, since the one at $x_2 \approx -7.9$ in Figure 3 moves to $-\infty$ for $r \to 0$ and is not relevant. Hence the gain-margins are equal to $k_- = 0$ and $k_+ \approx -1/(-0.54) \approx 1.85$. Similar we can read off the crossing point of the Nyquist plot with the unit circle which is closest to -1 in order to obtain the phase-margin $360\frac{\arctan(0.53/0.85)}{2\pi} \approx 31.9$ degrees. With the command `allmargin(L)` in Matlab one extracts the complete information about the points of crossings and the corresponding frequencies.

The relevance of phase-perturbations for loop transfer functions is motivated by consid-

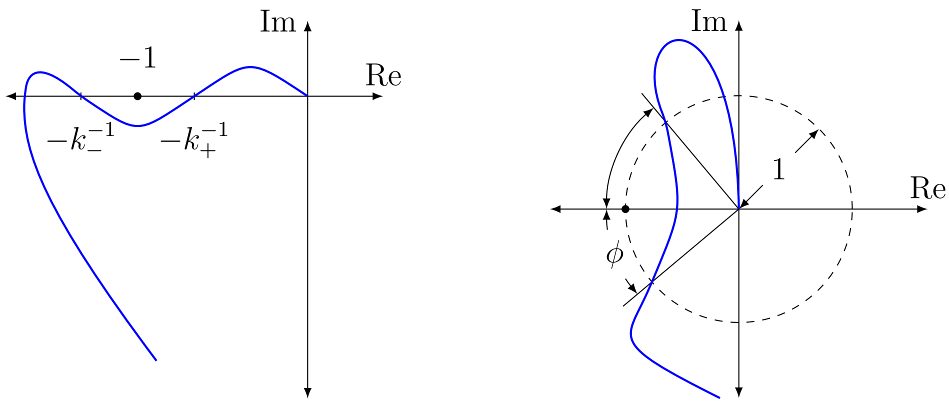


Figure 4: Graphical illustration of gain-margins and phase-margin.

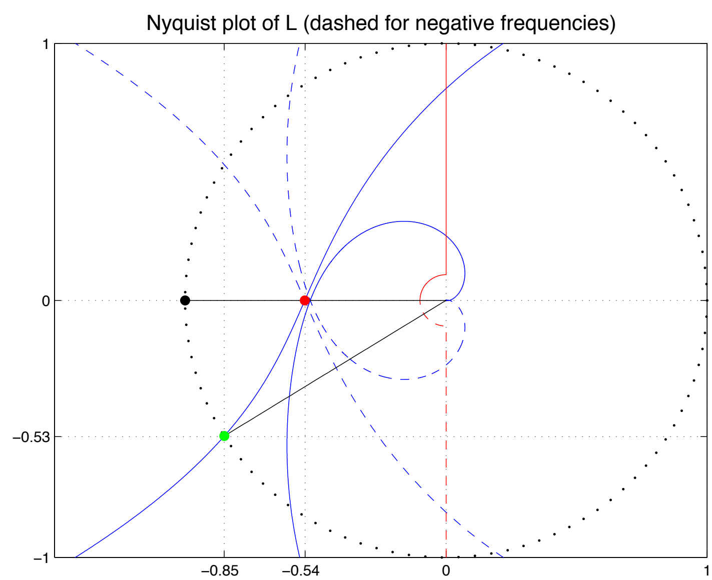


Figure 5: Nyquist plot with unit circle (dotted) and relevant crossing points.

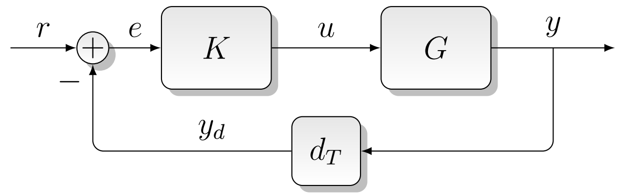


Figure 6: Standard tracking configuration with time-delay.

ering time-delays in feedback loops. A **time-delay** d_T of T seconds takes $u(\cdot)$ into

$$y(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t < T, \\ u(t-T) & \text{for} \quad t \geq T \end{cases}$$

if t is measured in seconds. Now recall that

$$\hat{y}(s) = \int_0^\infty e^{-st} y(t)\, dt = \int_T^\infty e^{-st} u(t-T)\, dt = \int_0^\infty e^{-s(\tau+T)} u(\tau)\, d\tau =$$
$$= e^{-sT} \int_0^\infty e^{-s\tau} u(\tau)\, d\tau = e^{-sT} \hat{u}(s).$$

Hence the transfer function of a time-delay of T seconds is given by e^{-sT}.

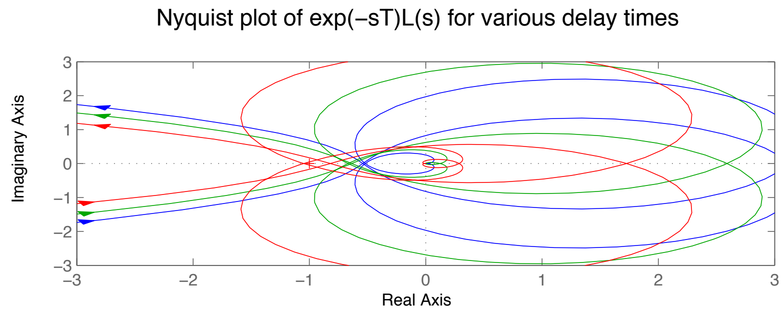
Even if such an element, which is not a real rational function, appears in a feedback-loop, the Nyquist stability criterion stays valid! To model a delay in the measurement of the system output by T seconds we consider the interconnection in Figure 6, which shows that we have to now consider the loop transfer function $\tilde{L}_T(s) = e^{-sT} L(s)$ with $L(s) = G(s)K(s)$ to analyze stability. If starting from some stable $(1+L)^{-1}$, we can ask when stability is lost if continuously increasing the delay time T.

This motivates to define the **time-delay-margin** T_m, which is the smallest $T \geq 0$ for which there exists a frequency $\omega \geq 0$ with

$$e^{-i\omega T} L(i\omega) = -1;$$

if no such T exists we set $T_m = \infty$. Suppose that $(1+L)^{-1}$ is stable and $L(\infty) = 0$. An application of the Nyquist criterion allows to conclude that $(1 + e^{-sT} L(s))^{-1}$ has all its poles in $\mathbb{C}_<$ for all $T \in [0, T_m)$, which means that the loop with all these delays stays stable.

Note that since $|e^{-i\omega T}| = 1$ for all $\omega \in \mathbb{R}$ and all $T \geq 0$, there is a close relationship between the phase and time-delay margin. Assuming that $|L(i\omega)| = 1$ only for finitely many $\omega \in \mathbb{R} \cup \{\infty\}$, one can compute the time-delay-margin as follows. Compute $\omega_1, \ldots, \omega_n$ where the Nyquist plot of L crosses the unit circle. Then solve $e^{-i\omega_j T_j} L(i\omega_j) = -1$ for $T_j \in [0, 2\pi]$ and all $1 \leq j \leq n$. Then the time-delay-margin equals $\inf_{1 \leq j \leq n} T_j$ if $n \geq 1$. If the Nyquist plot of L does not cross the unit circle, the time-delay-margin equals ∞.



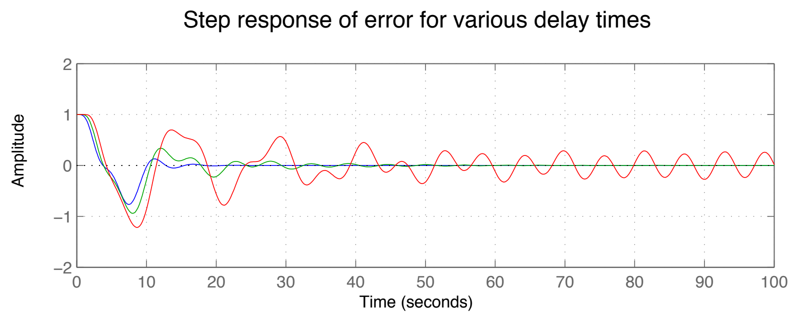


Figure 7: Responses of the system from Example 2.6 with time-delay $T \in \{0, 0.4, 0.9\}$ in the loop.

Example 2.8 The time-delay-margin for the transfer functions in Example 2.6 equals

$$T_m \approx 0.98\, s \approx \frac{2\pi}{360} \frac{60.93^\circ}{1.085 \frac{\text{rad}}{s}}.$$

Mind the units of the frequency in this computation! Some responses of the system affected by time-delay $T \in \{0, 0.4, 0.9\}$ are given in Figure 7.

In practice, gain-variations and time-delays might occur simultaneously. It is hence no longer sufficient to only consider real or complex unitary values of c if perturbing L to cL. Common perturbations are modeled by complex numbers c, whose size is measured in terms of their absolute value.

In view of the discussion so far, it is clear that the distance of the Nyquist plot of L from the point -1 is the correct stability margin; if the transfer function L has no poles in $\mathbb{C}_=$, this distance (for a Nyquist contour with $r = 0$ and $R = \infty$) just equals

$$\inf_{\omega \in \mathbb{R}} |L(i\omega) + 1|.$$

This distance is related to the sensitivity transfer function as follows.

Theorem 2.9 *For real-rational L without poles in $\mathbb{C}_= \cup \{\infty\}$ we have*

$$\left(\inf_{\omega \geq 0} |L(i\omega) + 1| \right)^{-1} = \sup_{\omega \geq 0} \left| \frac{1}{1 + L(i\omega)} \right|.$$

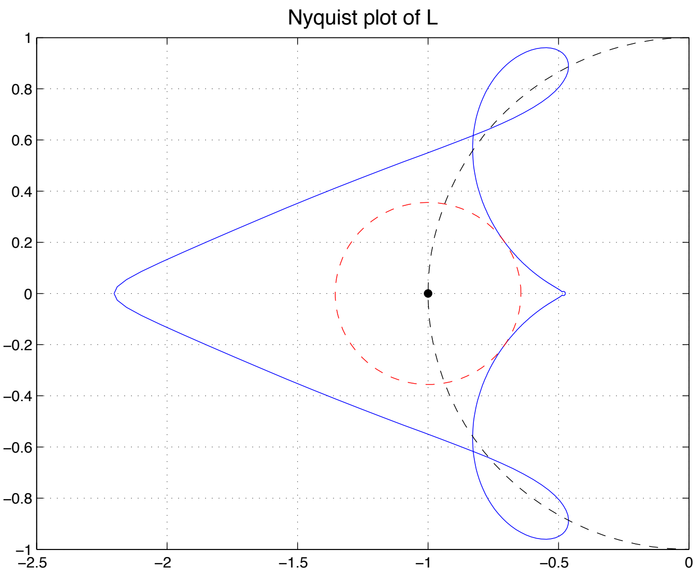


Figure 8: Nyquist plot of L in Example 2.10, unit circle and circle around -1 with radius $\inf_{\omega\in\mathbb{R}} |L(i\omega)+1|$.

This is a triviality. For a feedback-loop such that $(1+L)^{-1}$ is stable, it says that the peak-value of the sensitivity transfer function

$$\|S\|_\infty$$

is inversely proportional to the distance of the Nyquist plot to -1. Roughly speaking, the smaller $\|S\|_\infty$ is, the larger are the allowed common variations in gain and phase (measured by $|c|$ if replacing L with cL) without violating stability of the loop.

Example 2.10 Consider the Nyquist plot of

$$L(s) = \frac{-0.46832(s+3.3)(s-2)(s+0.55)(s^2+0.8824s+0.5882)}{(s+0.303)(s-0.5)(s+1.818)(s^2+1.5s+1.7)}$$

which is depicted in Figure 8. One can see that the gain- and phase margins are much larger than the distance of the Nyquist plot to -1. Hence the gain- and phase-margins provide a wrong guideline for robust stability if considering commonly occurring variations in gain and phase.

Up to this point we were rather sloppy in discussing robust stability and we did not give complete proofs for our statements. The remaining parts of these notes serve to develop a systematic theoretical basis for robustness questions that is also applicable to multivariable systems.

2.4 Plant Uncertainty

The open-loop plant G is obtained by physical or experimental modeling. If the actual physical system is described by H, we are typically confronted with a plant-model mismatch, which just means $G \neq H$. If we design a controller for G, it should as well do a good job for H, even if H deviates from G not only slightly but even significantly.

The general philosophy of robust control can be expressed as follows: Instead of designing good controllers for just one model G of H, one rather considers a whole set of models $\mathcal{H}$ with $H \in \mathcal{H}$ and designs a controller which does a good job for all elements in $\mathcal{H}$.

A mathematical description of $\mathcal{H}$ is a so-called uncertainty model. Whether or not $H \in \mathcal{H}$ is true, is a question of validating the uncertainty model and part of the field of system identification.

Concretely, let us measure the deviation of G and H at some frequency ω with $G(i\omega) \neq 0$ by the relative error of their frequency responses:

$$\frac{|H(i\omega) - G(i\omega)|}{|G(i\omega)|} = \left|\frac{H(i\omega)}{G(i\omega)} - 1\right|.$$

In practice, this error is typically small at low frequency and large at high frequencies. The variation of the size of this error over frequency is captured with a transfer function W by requiring

$$\left|\frac{H(i\omega)}{G(i\omega)} - 1\right| < |W(i\omega)| \ \text{ if } \ W(i\omega) \neq 0 \ \text{ and } \ H(i\omega) = G(i\omega) \ \text{ otherwise.}$$

Note that this holds iff there exists some $\Delta_\omega \in \mathbb{C}$ with

$$H(i\omega) = G(i\omega)(1 + W(i\omega)\Delta_\omega) \quad \text{and} \quad |\Delta_\omega| < 1.$$

Here G is the **nominal system** and W is a so-called **uncertainty weight** which captures the size of the deviation of $H(i\omega)$ from $G(i\omega)$. This is said to be a **multiplicative uncertainty model.**

The mathematically precise definition is based on a **nominal model** described by a transfer function $G \neq 0$ and a **weight** described by a stable transfer function $W \neq 0$.

Definition 2.11 *The* multiplicative uncertainty model $\mathcal{H}$ *related to a transfer function G and a stable transfer function W is the set of all transfer functions H such that*

1) H and G have the same number of poles in $\mathbb{C}_>$,

2) H and G have identical poles on the imaginary axis $\mathbb{C}_=$ and

3) $|H(i\omega)/G(i\omega) - 1| < |W(i\omega)|$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ with $W(i\omega) \neq 0$.

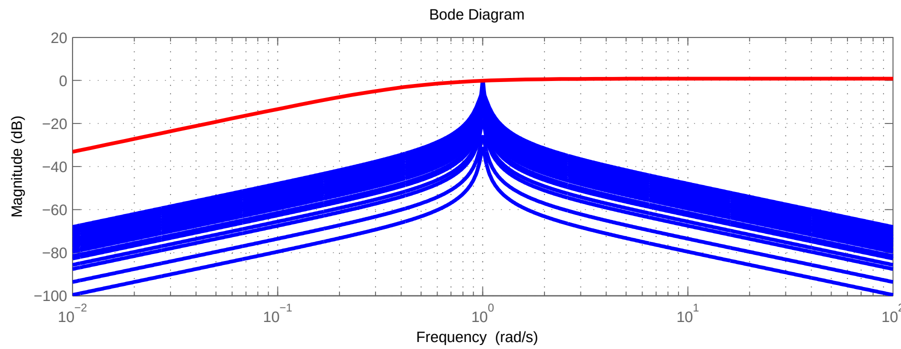


Figure 9: Graphical verification of 3) in Definition 2.11.

We emphasize that H/G does not have poles in $\mathbb{C}_= \cup \{\infty\}$ due to 3). Also note that, in practice, W typically is a **high-pass filter**.

Example 2.12 This continues Example 2.6 and 2.7. Suppose that some parameter in our satellite model is not known exactly, and that is transfer function is actually given by

$$G_b(s) = \frac{0.036(s+25.28)}{s^2(s^2+bs+1)} \quad \text{for some} \quad b \in (\tfrac{1}{2}b_0, 2b_0) \quad \text{with} \quad b_0 = 0.0396.$$

Choose $W(s) = 1.1s/(s+0.5)$ to obtain

$$\left|\frac{G_b(i\omega)}{G_{b_0}(i\omega)} - 1\right| < |W(i\omega)| \quad \text{for all} \quad \omega \in \mathbb{R} \quad \text{with} \quad \omega \neq 0,$$

as verified graphically through the plot in Figure 9. With G_{b_0} as the nominal model and W as the uncertainty weight, define the uncertainty model $\mathcal{H}$. Since all G_b as described above have $s = 0$ as a double pole in $\mathbb{C}_= \cup \mathbb{C}_>$ we indeed infer $G_b \in \mathcal{H}$.

It is emphasized that $\mathcal{H}$ contains many more models. In that sense we "cover" or "over-bound" the actual uncertainty in the parameter b through an uncertainty model that also comprises dynamic uncertainties. All models

$$G_{b_0}(s)(1 + W(s)\Delta(s)) = \frac{0.036(s+25.28)}{s^2(s^2+0.0396s+1)}\left(1 + \frac{1.1s}{s+.5}\Delta(s)\right)$$

for arbitrary $\Delta \in RH_\infty$ satisfying $\|\Delta\|_\infty < 1$ belong to $\mathcal{H}$ as well:

Since Δ cannot add poles in $\mathbb{C}_>$ property 1) is still satisfied. Since $1 + W(0)\Delta(0) = 1$, we also observe that the double pole of G_{b_0} at $s = 0$ is never canceled; hence 2) holds. The property 3) is satisfied because $\|\Delta\|_\infty < 1$ implies $|W(i\omega)\Delta(i\omega)| < |W(i\omega)|$ for $\omega \in \mathbb{R} \cup \{\infty\}$ with $\omega \neq 0$.

For the designed controller K and models $H \in \mathcal{H}$, samples of the Bode magnitude and step response plots are depicted in Figure 10. We conclude that K does not robustly

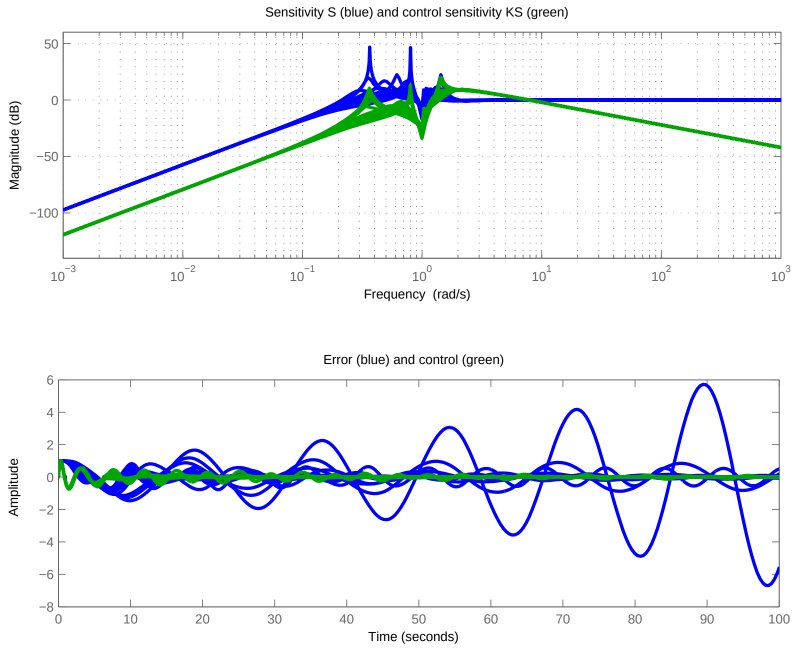


Figure 10: Samples of the Bode magnitude and step response.

stabilize $\mathcal{H}$. This just means there exists some $H \in \mathcal{H}$ for which $(1+HK)^{-1}$ is not stable.

Theorem 2.13 *Let $\mathcal{H}$ be given as in Definition 2.11. Suppose the transfer function K does not cause any pole-zero cancellation in $\mathbb{C}_= \cup \mathbb{C}_>$ if forming GK and renders $(1+GK)^{-1}$ proper and stable. Then*

$$(1+HK)^{-1} \quad \textit{exists and is proper and stable for all} \quad H \in \mathcal{H} \tag{2.3}$$

if and only if

$$\|WGK(1+GK)^{-1}\|_\infty \leq 1. \tag{2.4}$$

The property (2.3) is the **robust stability** question with respect to the multiplicative uncertainty model $\mathcal{H}$ under consideration.

The condition (2.4) provides an **exact verifiable robust stability test**: For the nominal model G, verify whether all unstable poles of G and K appear in $L = GK$ and whether $(1+L)^{-1}$ is proper and stable. If this is the case, then just determine the complementary sensitivity transfer function $T = L(1+L)^{-1} = GK(1+GK)^{-1}$ and check $\|WT\|_\infty \leq 1$.

With $L = GK$ we note that (2.4) reads as $\|WL(1+L)^{-1}\|_\infty \leq 1$; if L has no poles in $\mathbb{C}_=$ then this condition has also a nice graphical interpretation. For this purpose observe

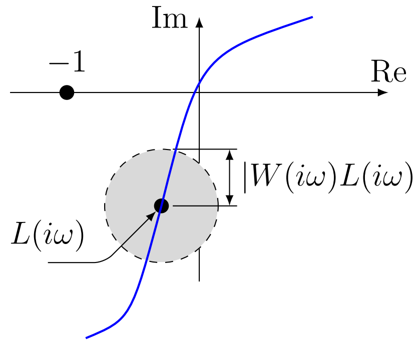


Figure 11: Graphical interpretation of (2.4).

that (2.4) can be expressed as

$$|W(i\omega)L(i\omega)| \leq |-1-L(i\omega)| \text{ for all } \omega \in \mathbb{R}\cup\{\infty\}. \tag{2.5}$$

For each frequency, this means that the distance of $L(i\omega)$ to -1 is not smaller than $|W(i\omega)L(i\omega)|$. We can equivalently say that the open disc $D_\omega = \{\lambda \in \mathbb{C}|\ |\lambda - L(i\omega)| < |W(i\omega)L(i\omega)|\}$ does not contain -1. This is depicted in Figure 11

Proof. "$\Leftarrow$": Choose any $H \in \mathcal{H}$ and define $\Delta := \frac{1}{W}(\frac{H}{G}-1)$. With 3) in Definition 2.11 we infer that

$$|\Delta(i\omega)| < 1 \text{ for all } \omega \in \mathbb{R}\cup\{\infty\}. \tag{2.6}$$

Hence Δ is a transfer function without any poles in $\mathbb{C}^0$. Now define

$$L_\tau := (1+\tau\Delta W)GK \text{ for } \tau \in [0,1].$$

Since $1+\Delta W = H/G$ we infer $L_0 = GK$ and $L_1 = HK$. We know that $(1+L_0)^{-1}$ is stable and we need to show stability of $(1+L_1)^{-1}$. This will be done by applying the Nyquist stability criterion and on the basis of the following key relation: By the very definition we have $L_\tau + 1 = (L_0+1) + \tau\Delta W L_0$ and thus

$$S(L_\tau+1) = 1+\tau\Delta W T. \tag{2.7}$$

for the stable transfer functions $S = (1+L_0)^{-1}$ and $T = 1-S = L_0(1+L_0)^{-1}$.

Let us choose $R>0$, $r>0$ defining a Nyquist contour Γ for G, H, K, W and Δ; so all poles of these transfer functions in $\mathbb{C}_= \cup \mathbb{C}_>$ and none in $\mathbb{C}_<$ are encircled by Γ. Note that Γ is also a Nyquist contour for all L_τ with $\tau \in [0,1]$. We recall again that these properites are not altered by increasing $R>0$ and decreasing $r>0$.

With (2.6) and $\|WT\|_\infty \leq 1$ we infer $|\Delta(\lambda)W(\lambda)T(\lambda)| < 1$ for all $\lambda \in \mathbb{C}_= \cup \{\infty\}$; if applied for $\lambda = \infty$, we see that we can increase the given $R>0$ in order to also ensure

$$|\Delta(\lambda)W(\lambda)T(\lambda)| < 1 \text{ for all } \lambda \in \mathbb{C}_= \cup \mathbb{C}_> \text{ with } |\lambda| \geq R; \tag{2.8}$$

for reasons of continuity, we can decrease $r > 0$ to also guarantee $|\Delta(\lambda)W(\lambda)T(\lambda)| < 1$ for all points $\lambda \in \Gamma$ on the indented imaginary axis; taken together this implies

$$|\Delta(\lambda)W(\lambda)T(\lambda)| < 1 \text{ for all } \lambda \in \Gamma. \tag{2.9}$$

After having fixed this Nyquist contour Γ for L_τ, we conclude from (2.7) and (2.9) that

$$L_\tau(\lambda) \neq -1 \text{ for all } \lambda \in \Gamma \text{ and all } \tau \in [0,1]; \tag{2.10}$$

otherwise we had $0 = |1+\tau\Delta(\lambda)W(\lambda)T(\lambda)| \geq 1-\tau|\Delta(\lambda)W(\lambda)T(\lambda)| > 0$, a contradiction. Let us now denote by $N(L_\tau)$ the number of counterclockwise encirclements of -1 of $L_\tau(\Gamma)$. Then (2.10) shows

$$N(L_0) = N(L_1). \tag{2.11}$$

Indeed, let $\gamma : [0,1] \to \Gamma$ be a (bijective piece-wise continuous differentiable) parametrization of Γ such that $\gamma(t)$ moves in clockwise direction around Γ if t moves from zero to one. It is then very easy to check that the map $H : [0,1] \times [0,1] \ni (\tau, t) \to L_\tau(\gamma(t)) \in \mathbb{C}$ is continuous: Choose any $(\tau_0, t_0) \in [0,1] \times [0,1]$; since Γ is compact and does not pass through any pole of Δ, W and $L_0 = GK$, there exists some constant b with $|\Delta(\gamma(t))W(\gamma(t))L_0(\gamma(t))| \leq b$ for all $t \in [0,1]$; for $(\tau,t) \in [0,1] \times [0,1]$ we get

$$\begin{aligned}
|L_\tau(\gamma(t)) - L_{\tau_0}(\gamma(t_0))| &\leq |L_\tau(\gamma(t)) - L_{\tau_0}(\gamma(t))| + |L_{\tau_0}(\gamma(t)) - L_{\tau_0}(\gamma(t_0))| \leq \\
&\leq |\tau - \tau_0||\Delta(\gamma(t))W(\gamma(t))L_0(\gamma(t))| + |L_{\tau_0}(\gamma(t)) - L_{\tau_0}(\gamma(t_0))| \leq \\
&\leq |\tau - \tau_0| b + |L_{\tau_0}(\gamma(t)) - L_{\tau_0}(\gamma(t_0))| \to 0 \text{ for } (\tau, t) \to (\tau_0, t_0).
\end{aligned}$$

As a well-known fact in complex analysis, that already this fact allow us to conclude that $N(L_\tau)$ is constant for $\tau \in [0,1]$, which in turn clearly leads to (2.11).

We are now ready to conclude the proof with the Nyquist stability criterion. First, since Γ is a Nyquist contour for L_0, L_1, their Nyquist plots do not pass through -1. Second, from (2.7) and (2.8), we infer $L_\tau(\lambda) \neq -1$ for all $\lambda \in \mathbb{C}_= \cup \mathbb{C}_>$ with $|\lambda| > R$ and $\tau = 0, 1$. If $n_{0+}(g)$ denotes the number of poles of g in $\mathbb{C}^0 \cup \mathbb{C}^+$, Theorem 2.5 applied to L_0 shows

$$N(L_0) = n_{0+}(L_0).$$

By the non-cancellation hypothesis and 1), 2) in Definition 2.11 we conclude

$$n_{0+}(L_0) = n_{0+}(G) + n_{0+}(K) = n_{0+}(H) + n_{0+}(K) \geq n_{0+}(L_1). \tag{2.12}$$

Combined with (2.11) we infer $N(L_1) \geq n_{0+}(L_1)$ which concludes the proof.

"$\Rightarrow$": Set $M := WT$ and suppose $\|M\|_\infty > 1$. Then there exists some $\omega_0 \in [0, \infty]$ with $|M(i\omega)| > 1$. By continuity, we can make sure that $i\omega_0$ is not a pole of G and K. If defining $\delta_0 := -\frac{1}{M(i\omega_0)}$ we clearly have $|\delta_0| < 1$. If δ_0 is real set $\Delta(s) := \delta_0$ and otherwise construct a stable transfer function Δ as in Lemma 2.14. We infer $1+M(i\omega_0)\Delta(i\omega_0)=0$. With $H := G(1+W\Delta)$ we obtain

$$1+HK=(1+GK)(1+(1+GK)^{-1}GKW\Delta)=(1+GK)(1+M\Delta).$$

Since $1+GK$ has no pole at $i\omega_0$, we conclude that $1+HK$ has a zero at $i\omega_0$. Hence, $(1+HK)^{-1}$ does not exist, is not proper or is unstable. Moreover, for all $\omega \in \mathbb{R}\cup\{\infty\}$ with $W(i\omega)\neq 0$ we have

$$\left|\frac{H(i\omega)}{G(i\omega)}-1\right|=|\Delta(i\omega)||W(i\omega)|\leq|\delta_0||W(i\omega)|<|W(i\omega)|.$$

In this fashion we have constructed some H that is "almost" contained in $\mathcal{H}$ and which destabilizes $(1+HK)^{-1}$. As the only trouble, poles of G in $\mathbb{C}_= \cup \mathbb{C}_>$ might be canceled in the product $H=(1+\Delta W)G$ such that 1), 2) in Definition 2.11 are not valid.

To overcome this trouble let $p_1,\ldots,p_m \in \mathbb{C}_=\cup\mathbb{C}_>$ be those poles of G at which cancelation takes place; note that $p_j \neq \pm i\omega_0$. Since $1+W\Delta$ vanishes at these points, $W(p_j)\neq 0$ for $j=1,\ldots,m$. Lemma 2.15 shows that there exists a stable transfer function F satisfying

$$F(i\omega_0)=1,\quad |F(p_j)|<\frac{1}{|W(p_j)|}\ \text{ for }\ j=1,\ldots,m\ \text{ and }\ \|F\|_\infty\leq 1.$$

If we define $\tilde{H}=(1+[F\Delta]W)G$, then all properties above persist to hold; in addition, however, we infer

$$|1+F(p_j)\Delta(p_j)W(p_j)|\geq 1-|\delta_0||F(p_j)||W(p_j)|>0\ \text{ for }\ j=1,\ldots,m.$$

Hence $p_1,\ldots,p_m$ are not canceled any more in $(1+F\Delta W)G$ and thus $\tilde{H}\in\mathcal{H}$.

■

In the proof of Theorem 2.13 the following two lemmata were used.

Lemma 2.14 *Let $\omega_0>0$ and $\delta_0\in\mathbb{C}$. Choose*

$$\alpha=\pm|\delta_0|,\quad \beta=i\omega_0\,\frac{\alpha-\delta_0}{\alpha+\delta_0}.$$

Then the transfer function

$$\Delta(s)=\alpha\frac{s-\beta}{s+\beta}$$

is real-rational, proper and satisfies

$$\Delta(i\omega_0)=\delta_0\ \textit{ as well as }\ |\Delta(i\omega)|=|\delta_0|\ \textit{ for all }\ \omega\in\mathbb{R}.$$

Either for $\alpha=|\delta_0|$ or for $\alpha=-|\delta_0|$ the transfer function Δ is stable.

Proof. One can prove $\Delta_0 = \Delta(i\omega_0)$ by direct calculations. Since $|\alpha| = |\Delta_0|$, the vectors that correspond to the complex numbers $\alpha + \Delta_0$ and $\alpha - \Delta_0$ are perpendicular. Hence $\frac{\alpha-\Delta_0}{\alpha+\Delta_0}$ is purely imaginary. This implies that β is real. Therefore, the distances of $i\omega$ to β and to $-\beta$ are identical such that $|i\omega - \beta| = |i\omega + \beta|$ what implies $|\Delta(i\omega)| = |\alpha| = |\Delta_0|$. Moreover, a change of sign of α leads to the reciprocal of $\frac{\alpha-\Delta_0}{\alpha+\Delta_0}$ which, again due to the fact that this number is purely imaginary, changes the sign of the imaginary part. Hence we can adjust the sign of α to render β non-negative. Then Δ is stable. (Note that β might vanish what causes no problem!)

■

Lemma 2.15 *Let $\omega_0 > 0$, $p_1, \ldots, p_m \in (\mathbb{C}^0 \cup \mathbb{C}^+) \setminus \{-i\omega_0, i\omega_0\}$ located symmetrically with respect to the real axis and positive real values $\alpha_1, \ldots, \alpha_m$ be given. Then there exists a stable transfer function F such that*

$$F(i\omega) = 1, \ \|F\|_\infty \leq 1 \ \ \textit{and} \ \ |F(p_j)| < \alpha_j \ \ \textit{for} \ \ j = 1, \ldots, m.$$

Proof. For $\xi > 0$ and $\omega > 0$ define the stable transfer function

$$G_\xi(s) = \frac{2\xi\omega s}{s^2 + 2\xi\omega s + \omega^2}.$$

Then observe that $G_\xi(i\omega) = 1$, $|G_\xi(s)| < 1$ for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+$ with $s \neq \pm i\omega$ and $\frac{2\xi\omega s}{s^2+2\xi\omega s+\omega^2} \to 0$ for $\xi \to 0$ for fixed s with $s \neq \pm i\omega$ (in particular for $s = p_1, \ldots, p_m$). Hence there exist $\xi_1, \ldots, \xi_m$ such that $|G_{\xi_j}(p_j)| < \alpha_j$ for all $1 \leq j \leq m$, and this persists to hold if decreasing $\xi_j > 0$. Then $F := G_{\tilde{\xi}}$ with $\tilde{\xi} = \inf_{1 \leq j \leq m} \xi_j$ has the desired properties.

■

Remark 2.16 Let us represent $G = \frac{N_G}{D_G}$ and $K = \frac{N_K}{D_K}$ with coprime real numerator and denominator polynomials.

- Then S and T can be expressed as

  $$S = \frac{D_G D_K}{N_G N_K + D_G D_K} \quad \text{and} \quad T = \frac{N_G N_K}{N_G N_K + D_G D_K}.$$

  Here $N_G N_K + D_G D_K$ is the so-called **characteristic polynomial** of the feedback loop. Recall that S (and/or T) is stable iff $N_G N_K + D_G D_K$ has only zeros in $\mathbb{C}_<$.

- The non-cancellation hypothesis means that N_G, D_K and D_G, N_K have no common zeros in $\mathbb{C}_= \cup \mathbb{C}_>$. The relation of "non-cancellation" and internal stabilization will be clarified in Theorem 2.24.

- Under the hypotheses of Theorem 2.13, the proof shows that (2.4) also guarantees the absence of any pole-zero cancellations in $\mathbb{C}_= \cup \mathbb{C}_>$ for HK and any $H \in \mathcal{H}$!

Example 2.17 Consider

$$G(s) = \frac{1}{s-1}, \quad H(s) = \frac{1}{s-2} \ \text{ and } \ W(s) = \frac{1.1}{s+2}.$$

Then

$$\Delta(s) = \frac{1}{W(s)}\left(\frac{H(s)}{G(s)} - 1\right) = \frac{s+2}{1.1(s-2)}$$

satisfies $|\Delta(i\omega)| < 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$. Hence there exists a uncertainty model $\mathcal{H}$ with $H \in \mathcal{H}$. The controller $K \in \mathbb{R}$ (a static gain) renders $1/(1+GK)$ stable iff $K + (s-1)$ has all its roots in $\mathbb{C}_<$ iff $K > 1$. For $K = 3$ we get

$$\|WT\|_\infty = \left\|\frac{3.3}{(s+2)^2}\right\|_\infty = 0.825 \leq 1.$$

Hence K is guaranteed to render $1/(1+HK)$ stable by Theorem 2.13. Is that also true for $1/(1+H_\tau K)$ with

$$H_\tau = (1+\tau\Delta W)G \ \text{ for } \ \tau \in (0,1)?$$

Although one might be tempted to draw this conclusion, the answer is in general no. To see this, observe that $1/(1+H_\tau K) = (1+L_\tau)^{-1}$ with $L_\tau = (1+\tau\Delta W)GK = (1-\tau)GK + \tau HK$ defined as in the proof of Theorem 2.13. If $\tau \neq 0$ and $\tau \neq 1$ we can no longer guarantee that $n_{0+}(L_0) \geq n_{0+}(L_\tau)$ is true. It might very well happen that $n_{0+}(L_0) < n_{0+}(L_\tau)$ holds; with the Nyquist contour and the notation in the proof of Theorem 2.13 we get

$$N(L_\tau) = N(L_0) = n_{0+}(L_0) < n_{0+}(L_\tau),$$

which actually implies instability of $(1+L_\tau)^{-1}$ by Theorem 2.5.

Example 2.18 For the uncertainty model and transfer matrices in Example 2.12 and the earlier designed controller we have $\|WT\|_\infty > 1.5 > 1$, which confirms non-robustness as seen by our simulations. If we reduce the size of the uncertainty by replacing the weight through $\tilde{W}(s) = 0.8s/(s+0.5)$, the simulations in Figure 12 seem to confirm robust stability. However, this is misleading since $\|\tilde{W}T\|_\infty \approx 1.09 > 1$ ensures that there exists some system in the uncertainty model for which the sensitivity is **not** stable.

In general, the size of the uncertainty in our general uncertainty description can be scaled with a factor $r > 0$ by replacing 3) in Definition 2.11 through

$$|H(i\omega)/G(i\omega) - 1| < r|W(i\omega)| \ \text{ for } \ \omega \in \mathbb{R} \cup \{\infty\}, \ W(i\omega) \neq 0.$$

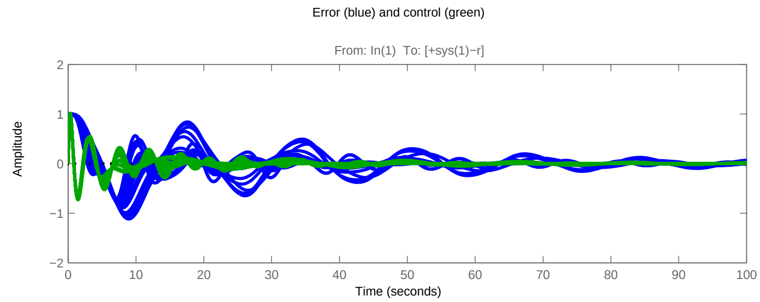


Figure 12: Sample plots for the rescaled uncertainty model $\mathcal{H}_r$ in Example 2.18.

For the corresponding uncertainty model $\mathcal{H}_r$ and under the hypothesis of Theorem 2.13 we then infer that

$$(1+HK)^{-1} \text{ exists and is proper and stable for all } H \in \mathcal{H}_r \tag{2.13}$$

if and only if $\|WGK(1+GK)^{-1}\|_\infty \leq \frac{1}{r}$. This gives a formula for the corresponding **robust stability margin**, the largest value of r for which (2.13) is valid.

Corollary 2.19 *Under the hypotheses of Theorem 2.13, $\|WGK(1+GK)^{-1}\|_\infty^{-1}$ is the maximal $r > 0$ for which (2.13) is satisfied.*

2.5 Performance

Consider again the interconnection depicted in Figure 1. Let $S = (1+L)^{-1}$ be stable with $L = GK$ and let (A, B, C, D) be a minimal realization of S. For a reference signal with constant value r, the error response equals

$$e(t) = Ce^{At}A^{-1}Br + (D - CA^{-1}B)r$$

and has the steady-state response $(D - CA^{-1}B)r = S(0)r$, since the transient response $Ce^{At}A^{-1}Br$ converges to 0 for $t \to \infty$ (just because S is stable, A is Hurwitz).

If $r(t) = r_0e^{i\omega t}$ is a complex sinusoidal signal with $r_0 \in \mathbb{C}$ and $\omega > 0$ we get

$$e(t) = Ce^{At}(A - i\omega I)^{-1}Br_0 + (C(i\omega I - A)^{-1}B + D)r_0e^{i\omega t}.$$

Again, the transient response $Ce^{At}(A - i\omega I)^{-1}Br_0$ converges (exponentially fast) to 0 for $t \to \infty$ due to the stability of S and minimality of the realization. Hence the steady-state response equals $(C(i\omega I - A)^{-1}B + D)r_0e^{i\omega t} = S(i\omega)r_0e^{i\omega t}$and is as well a sinusoidal signal with the same frequency as $r(.)$ and complex amplitude given by $e_0 = S(i\omega)r_0$; clearly $|S(i\omega)|$ is the amplification (or attenuation) factor for the amplitude of sinusoidal signals of frequency ω in $e = Sr$. Hence $|e(t)| \approx |S(i\omega)||r_0|$ for sufficiently large $t \geq 0$. Since

e is the tracking error for the configuration in Figure 1, sinusoidal reference signals of frequency ω are tracked well if $|S(i\omega)|$ is small. By

$$|S(i\omega)| = \frac{1}{|1+G(i\omega)K(i\omega)|} = \frac{1}{|K(i\omega)|}\frac{1}{|\frac{1}{K(i\omega)}+G(i\omega)|},$$

this can be achieved with large values of $|G(i\omega)|$ or of $|K(i\omega)|$ if $G(i\omega) \neq 0$. Due to $u = KSr$, the control action in the loop is related to $|K(i\omega)S(i\omega)| = |\frac{1}{K(i\omega)} + G(i\omega)|^{-1}$; this gain is large for large $|K(i\omega)|$ and small $|G(i\omega)|$. Since $|G(i\omega)|$ is often small for high frequencies, large values of $|K(i\omega)|$ imply

$$\frac{|G(i\omega)K(i\omega)|}{|1+G(i\omega)K(i\omega)|} = \frac{|G(i\omega)|}{|\frac{1}{K(i\omega)}+G(i\omega)|} \approx 1,$$

which, roughly, means that the relative plant-model mismatch cannot go beyond 100% at high frequencies in order not to endanger stability.

All this motivates that it is, in general, not a good design goal to try to reduce $|S(i\omega)|$ over **all** frequencies by control. Instead, in practice one chooses e.g. some frequency band $[\omega_1, \omega_2]$ and tries to suppress $|S(i\omega)|$ for $\omega \in [\omega_1, \omega_2]$ only.

Mathematically, this is modeled with a **performance weight** W_p, a stable transfer function that allows to express the desired specification precisely as

$$\|W_p S\|_\infty \leq 1.$$

If this inequality holds, this simply means that

$$|S(i\omega)| \leq \frac{1}{|W_p(i\omega)|} \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}. \tag{2.14}$$

If true, the tracking error is small "in the frequency range" $[\omega_1, \omega_2]$ if W_p had been taken such that $|W_p(i\omega)|$ is large at all these frequencies. In practice, W_p are **band-pass** filters. To track constant references we choose $\omega_1 = 0$ and take W_p as a **low-pass** filter.

The inequality (2.14) has again a nice geometric interpretation. If $i\omega$ is not a pole of L, then (2.14) can be written as

$$|W_p(i\omega)| \leq |1 + L(i\omega)| = |-1 - L(i\omega)|.$$

This means that $L(i\omega)$ should lie outside the open disk with center -1 and radius $|W_p(i\omega)|$ as depicted in Figure 13.

Example 2.20 This continues the Examples 2.6, 2.7 and 2.12. If we choose

$$W_p(s) = \left(\frac{s/\sqrt{M}+\omega_B}{s+\omega_B\sqrt{A}}\right)^2 = \left(\frac{s/\sqrt{5}+0.1}{s+0.1\sqrt{10^{-3}}}\right)^2$$

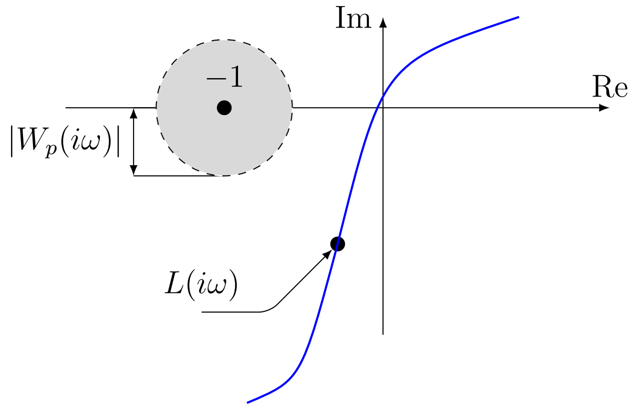


Figure 13: Geometric interpretation of (2.14).

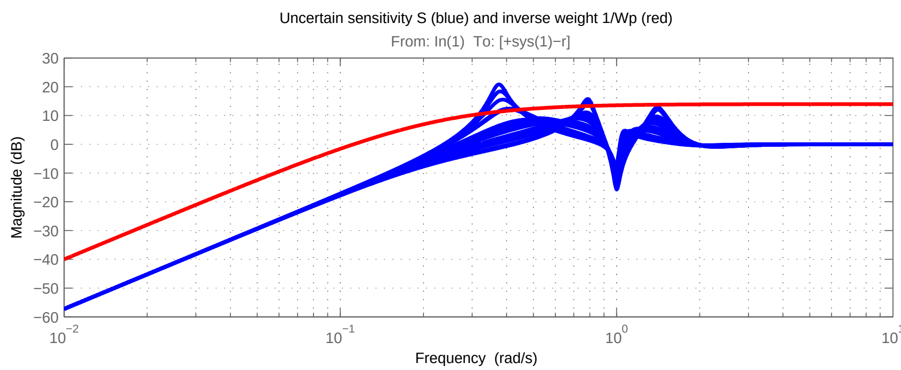


Figure 14: Samples of Bode magnitude plots.

one can check that $\|W_p(1+GK)^{-1}\| \leq 1$. If $\mathcal{H}$ is given for the uncertainty weight $W(s) = 0.7s/(s+0.5)$ we now ask whether the following **robust performance** specification holds:

$$\|W_p(1+HK)^{-1}\|_\infty \leq 1 \quad \text{for all} \quad H \in \mathcal{H}.$$

It is easy to check $\|WT\|_\infty \leq 1$ which implies robust stability. However, the samples of Bode magnitude plots in Figure 14 show that robust performance does not hold.

Theorem 2.21 *Let W_p be a stable transfer function and let the hypothesis of Theorem 2.13 be satisfied. Then robust stability (2.3) and robust performance as characterized by*

$$\|W_p(1+HK)^{-1}\|_\infty \leq 1 \quad \textit{for all} \quad H \in \mathcal{H} \tag{2.15}$$

are satisfied if and only if

$$|W_p(i\omega)S(i\omega)| + |W(i\omega)T(i\omega)| \leq 1 \quad \textit{for all} \quad \omega \in \mathbb{R} \cup \{\infty\} \tag{2.16}$$

holds with $S = (1+GK)^{-1}$ and $T = GK(1+GK)^{-1}$.

Again, the essence is an **easily verifiable** characterization of robust stability and robust performance in terms of the nominal system G, the controller K and the weights W and W_p capturing the uncertainty and the performance specification. Note that (2.16) imply

$$\|W_p S\|_\infty \leq 1 \quad \text{and} \quad \|WT\|_\infty \leq 1,$$

i.e., nominal performance and robust stability. Also note that (2.16) **cannot** be expressed as an H_∞-norm constraint; a sufficient condition for (2.16) can be expressed as

$$\left\| \begin{pmatrix} W_p S \\ WT \end{pmatrix} \right\|_\infty < \frac{1}{\sqrt{2}}$$

.

Proof. "$\Leftarrow$": (2.16) also implies $|W(i\omega)T(i\omega)| \leq 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$. Since by assumption $(1+GK)^{-1}$ and W are stable, this means $\|WT\|_\infty \leq 1$. Hence $(1+HK)^{-1}$ is stable for all $H \in \mathcal{H}$ by Theorem 2.13.

Suppose $i\omega \in \mathbb{C}^0$ is not a pole of G, K and W_p, W, $1/W$. Let $H \in \mathcal{H}$ and define $\Delta := \frac{1}{W}(\frac{H}{G} - 1)$. Due to 3) in Definition 2.11 we infer $|\Delta(i\omega)| < 1$. Also observe that $H = (\Delta W + 1)G$. Moreover we conclude from (2.16) that

$$|W_p(i\omega)| + |W(i\omega)G(i\omega)K(i\omega)| \leq |1 + G(i\omega)K(i\omega)|.$$

By using $|\Delta(i\omega)| < 1$ we obtain

$$\begin{aligned} |W_p(i\omega)| &\leq |1 + G(i\omega)K(i\omega)| - |W(i\omega)G(i\omega)K(i\omega)| \\ &\leq |1 + G(i\omega)K(i\omega)| - |\Delta(i\omega)||W(i\omega)G(i\omega)K(i\omega)| \\ &\leq ||1 + G(i\omega)K(i\omega)| - |\Delta(i\omega)W(i\omega)G(i\omega)K(i\omega)|| \\ &\leq ||1 + G(i\omega)K(i\omega) + \Delta(i\omega)W(i\omega)G(i\omega)K(i\omega)|| \\ &\leq |1 + [1 + \Delta(i\omega)W(i\omega)]\, G(i\omega)K(i\omega)| \\ &= |1 + H(i\omega)K(i\omega)| \end{aligned}$$

which shows $|W_p(i\omega)(1 + H(i\omega)K(i\omega))^{-1}| \leq 1$. Due to stability of W_p and $(1+HK)^{-1}$ and continuity in ω, this inequality holds in fact for all $\omega \in \mathbb{R} \cup \{\infty\}$. Hence (2.15) follows.

"$\Rightarrow$": If $|W_p(i\tilde{\omega})S(i\tilde{\omega})| + |W(i\tilde{\omega})T(i\tilde{\omega})| > 1$ for some $\tilde{\omega} \in \mathbb{R} \cup \{\infty\}$, we can also find $\omega_0 \in \mathbb{R}$ such that $i\omega_0$ is not a pole of G and K and, still, $|W_p(i\omega_0)S(i\omega_0)| + |W(i\omega_0)T(i\omega_0)| > 1$.

If $|W(i\omega_0)T(i\omega_0)| > 1$, robust stability is not guaranteed by Theorem 2.13 and the proof is finished. Otherwise we get $|W_p(i\omega_0)S(i\omega_0)| > 1 - |W(i\omega_0)T(i\omega_0)| \geq 0$ and thus

$$|W_p(i\omega_0)| > |1 + L(i\omega_0)| - |W(i\omega_0)L(i\omega_0)| \geq 0.$$

Lemma 2.22 for $u = 1 + L(i\omega_0)$ and $v = W(i\omega_0)L(i\omega_0)$ allows us to choose $\delta_0 \in \mathbb{C}$ with $|\delta_0| < 1$ and

$$|W_p(i\omega_0)| > |1 + L(i\omega_0) + \delta_0 W(i\omega_0)L(i\omega_0)| = |1 + [1 + \delta_0 W(i\omega_0)]G(i\omega_0)K(i\omega_0)|. \quad (2.17)$$

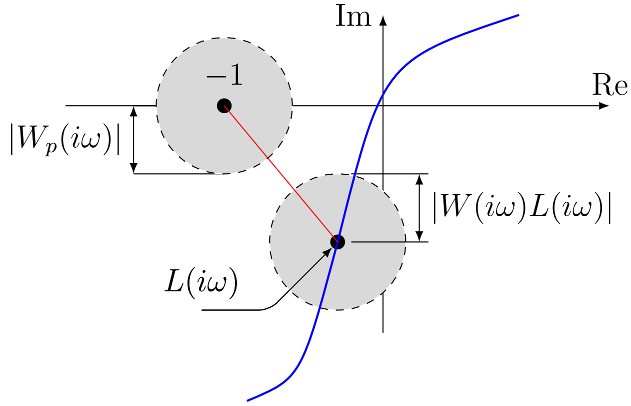


Figure 15: Geometric interpretation of (2.16).

If constructing $F\Delta \in RH_\infty$ such that $H := (1 + F\Delta W)G \in \mathcal{H}$ and $F\Delta(i\omega_0) = \delta_0$ as in the proof of Theorem 2.13, we infer that $i\omega_0$ is not a pole of H (since it was not a pole of G) and $|W_p(i\omega_0)| > |1 + H(i\omega_0)K(i\omega_0)|$ by (2.17). This shows $\|W_p(1 + HK)^{-1}\|_\infty > 1$.

■

Lemma 2.22 *Let us be given $\alpha \in \mathbb{R}$, $u, v \in \mathbb{C}$ with $\alpha > |u| - |v| \geq 0$. Then there exist $\delta_0 \in \mathbb{C}$ with $|\delta_0| < 1$ such that $\alpha > |u + \delta_0 v|$.*

Proof. There exist $\phi, \psi \in [-\pi, \pi]$ with $u = |u|e^{i\phi}$ and $v = |v|e^{i\psi}$. Define $\delta := e^{i(\phi-\psi+\pi)}$ to obtain

$$|u + \delta v| = ||u|e^{i\phi} + |v|e^{i\phi+i\pi}| = ||u|e^{i\phi} - |v|e^{i\phi}| = ||u| - |v|| = |u| - |v|.$$

Hence $\alpha > |u + \delta v|$; since the inequality is strict, there exists some small $\epsilon > 0$ such that $\alpha > |u + (1 - \epsilon)\delta v|$ and we can choose $\delta_0 := (1 - \epsilon)\delta$.

■

If $i\omega$ is not a pole of L, (2.16) can be written at this frequency as

$$|W_p(i\omega)| + |W(i\omega)L(i\omega)| \leq | - 1 - L(i\omega)|.$$

This means that the *distance* of $L(i\omega)$ to -1 is at least $|W_p(i\omega)| + |W(i\omega)L(i\omega)|$, i.e., the two open disks related to the nominal performance specification and the robust stability condition as shown in Figure 15 should not intersect.

2.6 Internal Stability

For transfer functions G and K let us now consider the feedback loop as depicted in Figure 16. So far we have concentrated on controllers K which render the transfer function

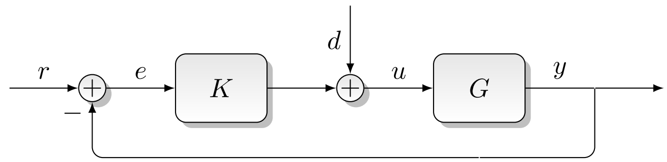


Figure 16: Standard tracking configuration with disturbance d.

in $e = (1+GK)^{-1}r$ stable. If true, this does not always imply that the transfer functions in $u = K(1+GK)^{-1}e$ or $y = G(1+GK)^{-1}d$ are stable. This motivates the following definition.

Definition 2.23 *K renders the feedback interconnection shwon in Figure 16* **internally stable** *if $(1+GK)^{-1}$ exists, is proper and if*

$$\frac{1}{1+GK}, \quad \frac{K}{1+GK}, \quad \frac{G}{1+GK}$$

are all stable.

Let us emphasize the following two useful facts:

- Stability of $(1+GK)^{-1}$ implies stability of

$$GK(1+GK)^{-1} = 1 - (1+GK)^{-1}.$$

- If $1+GK$ is not the zero transfer function we know that

$$\begin{pmatrix} 1 & G \\ -K & 1 \end{pmatrix}^{-1} = \frac{1}{1+GK}\begin{pmatrix} 1 & -G \\ K & 1 \end{pmatrix} = \begin{pmatrix} S & -GS \\ KS & S \end{pmatrix}.$$

The latter observation justifies the following alternative characterization of the conditions in Definition 2.23: K renders the feedback interconnection internally stable iff

$$\begin{pmatrix} 1 & G \\ -K & 1 \end{pmatrix}^{-1} \text{ exists and is stable.}$$

Theorem 2.24 *Suppose that $G = \frac{N_G}{D_G}$ and $K = \frac{N_K}{D_K}$ with coprime real numerator and denominator polynomials. Then K renders the feedback interconnection internally stable iff one of the following equivalent conditions holds:*

1) *There is no pole-zero cancellation in $\mathbb{C}_= \cup \mathbb{C}_>$ if forming GK and $(1+GK)^{-1}$ exists and is stable.*

2) The 2×2 *transfer matrix* $\begin{pmatrix} 1 & G \\ -K & 1 \end{pmatrix}$ *has a stable inverse.*

3) $1 + G(\infty)K(\infty) \neq 0$ *and the zeros of the characteristic polynomial* $N_G N_K + D_G D_K$ *of the loop are contained in* $\mathbb{C}_<$*.*

Recall that "stable" includes the requirement of properness. We will not emphasize this point in the sequel.

Proof. Define $p := N_G N_K + D_G D_K$. If $1 + GK \neq 0$ then

$$\begin{pmatrix} 1 & G \\ -K & 1 \end{pmatrix}^{-1} = \begin{pmatrix} S & -GS \\ KS & S \end{pmatrix} = \frac{1}{p} \begin{pmatrix} D_G D_K & -N_G D_K \\ D_G N_K & D_G D_K \end{pmatrix}. \tag{2.18}$$

1) $\Rightarrow$ 3). Since $S = (1 + GK)^{-1}$ is proper we get $1 + G(\infty)K(\infty) \neq 0$. Suppose $p(\lambda) = 0$ for $\lambda \in \mathbb{C}^0 \cup \mathbb{C}^+$. Since $S = D_G D_K / p$ is stable we infer $D_G(\lambda) D_K(\lambda) = 0$ and hence $N_G(\lambda) N_K(\lambda) = N_G(\lambda) N_K(\lambda) + D_G(\lambda) D_K(\lambda) = p(\lambda) = 0$. Then either $N_G(\lambda) = 0$ or $N_K(\lambda) = 0$.

If $N_G(\lambda) = 0$ then $D_G(\lambda) \neq 0$ since N_G and D_G are coprime. Hence $D_K(\lambda) = 0$ is true. This means that λ is a common zero of N_G and D_K that is cancelled when forming GK. This is a contradiction to 1).

An analogous reasoning leads to a contradiction if $N_K(\lambda) = 0$.

3) $\Rightarrow$ 2). Since $1 + G(\infty)K(\infty) \neq 0$, we have $1 + GK \neq 0$ and the inverse in (2.18) exists as a real rational function. Since $S = (1 + GK)^{-1}$, G and K are proper, the first equation in (2.18) shows that the inverse is proper. Since p has only zeros in $\mathbb{C}_<$, the inverse is stable as seen by the second equation in (2.18).

2) $\Rightarrow$ 1). Since the inverse exists we infer $1 + GK \neq 0$ as a function. Properness of the inverse and (2.18) show that $S = (1 + GK)^{-1}$ exists and is proper.

Let $\lambda \in \mathbb{C}_= \cup \mathbb{C}_>$ satisfy $N_G(\lambda) = 0$ and $D_K(\lambda) = 0$. On the one hand we clearly have $p(\lambda) = 0$. On the other hand, we know $D_G(\lambda) \neq 0$ (since N_G, D_G are coprime) and $N_K(\lambda) \neq 0$ (since N_K, D_K are coprime). Hence $D_G N_K / p$ has a pole at λ, which contradicts the stability of (2.18).

We infer that N_G, D_K have no common zeros in $\mathbb{C}_= \cup \mathbb{C}_>$, and analogous arguments show the same for D_G, N_K. We conclude that no $\mathbb{C}_= \cup \mathbb{C}_>$ pole-zero cancelation occurs in GK.

$\blacksquare$

With the concept of internal stabilization and in view of the hypothesis of Theorem 2.13 as well as the third bullet in Remark 2.16, we can rephrase Theorem 2.13 as follows: Suppose K internally stabilizes the interconnection in Figure 16 with G. Then K internally stabilizes this interconnection with any $H \in \mathcal{H}$ if and only if (2.4) holds true.

This sets the stage for the generalization to much more complicated general interconnections and structured uncertainties in the next chapters.

Exercises

1) Suppose you are given the system and controller

$$G = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline \alpha & -1 & 0 \end{array}\right] \text{ and } K = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right] \text{ with } \alpha = 1.$$

   a) Compute GK and $S = (1 + GK)^{-1}$ in the state-space by standard formulas for the product and the inverse of realizations.

   b) Is the resulting system-matrix Hurwitz?

   c) Does K render S stable? Is GS stable? Is KS stable?

   d) What is the trouble if α slightly deviates from 1?

2) For given transfer functions G and K let us consider the feedback interconnection in Figure 16.

   a) If GK is not equal to -1, show that the transfer matrix from the the reference and the input disturbance $\begin{pmatrix} r \\ d \end{pmatrix}$ to the signal $\begin{pmatrix} e \\ u \end{pmatrix}$ is given by

$$\begin{pmatrix} 1 & G \\ -K & 1 \end{pmatrix}^{-1}. \tag{2.19}$$

   Express the inverse in terms of S, G and K.

   b) For a given SISO system G, the SISO controller K is said to internally stabilize the standard feedback configuration of Figure 16 if (2.19) is stable. Construct an example such that K renders (2.19) stable except for the $(1;2)$- entry.

   c) Suppose G is stable. Then show that K internally stabilizes the tracking configuration iff K renders $K(1 + GK)^{-1}$ is stable.

3) Let L be a proper transfer function and Γ a Nyquist contour fo L with $R = \infty$ such that the Nyquist plot of L does not pass through -1 and $L(\infty) \neq -1$. Suppose

that L has n_{0+} poles on the imaginary axis or in the open right-half plane (counted with multiplicities). Show that $(1+L)^{-1}$ is stable if and only if the Nyquist plot of L encircles -1 exactly n_{0+} times in the counterclockwise direction.

4) In 195x NASA launched a flexible satellite in order to spy on the Russians. Based on the linear model $G(s) = \frac{0.036(s+25:28)}{s^2(s^2+0.00396s+1)}$ a team of control engineers designed and tested a linear controller $K(s) = \frac{7.9212(s+0:1818)(s^2-0.2244s+0.8981)}{(s^2+3.899s+4.745)(s^2+1.039s+3.395)}$ that performed sufficiently well for the job to be done. Now that the cold war is over, NASA found another purpose for the satellite. Unfortunately, due to a recent defect in the control hardware, the measurement got delayed by maximally $T = 0.25$ seconds. For this purpose, a robustness analysis has been scheduled in order to verify whether the system still performs sufficiently well, or whether a new controller has to be designed and uploaded. Unfortunately, NASA ran out of money, due to jet another fiscal cliff. For this reason NASA asked the University of Stuttgart (which is well-known for their excellent students) whether the job could be done by a student free of charge.

   Consider the time-delayed feedback interconnection depicted in Figure 6.

   a) Design a generic dynamical weight W_T of order 1 that "tightly covers" $d_T - 1$ with the delay operator d_T and its transfer function e^{-sT} and for arbitrary $T > 0$. "Covering" is interpreted as in Example 2.12.

   b) For the nominal model G and the weight W_T, show that the delayed transfer function H satisfies all properties in Definition 2.11.

   c) Perform a robustness analysis based on Theorem 2.13 and answer the following questions:

      i) Does the system remain stable?

      ii) What is the robust stability margin?

      iii) What is the maximum tolerable delay-time T for which $\|W_T(1+GK)^{-1}\|_\infty \leq 1$?

   d) What would be your recommendation for NASA?

   Hint: Although not proven rigorously, you can use the fact that (2.16) $\Rightarrow$ (2.15) in Theorem 2.13 stays valid for the delayed transfer function H. Use Matlab for solving this exercise!

5) This continues Exercise 4) with $\mathcal{H}$ for the nominal system G and the uncertainty weight W_T.

   a) Recall the first order weight W_T from Exercise 4) and the second order performance weight W_p from Example 2.20 and verify whether the controller achieves robust performance for a delay-time of $T = 0.25$ seconds.

b) What is the maximum tolerable time-delay $T_{\max}$ for which the controller still achieves robust performance?

c) For a time-delay of $T = 0.25$ seconds, what is the maximum bandwidth ω_B in the weight W_p for which the controller still achieves robust performance?

d) Let W_p again be as in Example 2.20 and choose a time-delay $T > T_{\max}$ with $|W_p(i\omega)S(i\omega)| + |W_T(i\omega)T(i\omega)| > 1$ for some $\omega \in \mathbb{R}$. Determine some $H \in \mathcal{H}$ such that $\|W_p(1 + HK)^{-1}\|_\infty > 1$.
HINT: Use Lemma 2.22.

6) For any transfer function Δ define an uncertain system through

$$G_\Delta(s) = \frac{1}{s}(1 + \Delta(s)).$$

The nominal system is $G_0(s) = 1/s$ and the controller is taken to be $K(s) = 1$. As usual define $L_\Delta = G_\Delta K$ and $T := L_0(1 + L_0)^{-1}$.

a) Show that T is stable and $\|T\|_\infty = 1$.

b) If Δ is a stable transfer function which satisfies $\Delta(0) = -1$, its Taylor expansion reads as $\Delta(s) = -1 + a_1 s + a_2 s^2 + \ldots$ around $s = 0$. If $\|\Delta\|_\infty \leq 1$ show that $a_1 \geq 0$.

c) Show that $(1 + L_\Delta)^{-1}$ is stable for all stable Δ satisfying $\|\Delta\|_\infty \leq 1$.

The next few exercises are more mathematical orientated.

7) This continues Exercise 2) with transfer functions G and K.

a) Show that K internally stabilizes G iff $(1 + GK)^{-1}$ is stable and no poles of G and K in $\mathbb{C}^0 \cup \mathbb{C}^+$ are canceled if forming GK.

b) Prove the Nyquist criterion for internal stability: Suppose m_{0+} is the number of poles of G and of K in $\mathbb{C}^0 \cup \mathbb{C}^+$ (including multiplicities). Then K internally stabilizes the tracking configuration iff the Nyquist plot of GK does not pass through -1 and encircles -1 exactly m_{0+} times in the counterclockwise direction.

8) Prove the following statements.

a) A continuous function $f : \mathbb{C}^0 \cup \mathbb{C}^+ \to \mathbb{R}$ is sub-harmonic if it satisfies, for all $z \in \mathbb{C}^+$ and all $r > 0$ with $r < \mathrm{Re}(z)$, the inequality

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})dt.$$

If $K \subset \mathbb{C}^0 \cup \mathbb{C}^+$ is a compact subset with boundary ∂K, show that

$$\sup_{z \in \partial K} f(z) = \sup_{z \in K} f(z).$$

b) For any matrix $M \in \mathbb{C}^{k\times m}$ show that $\|M\| = \max_{\|x\|=1,y\|=1} |x^* M y|$.

c) For $G \in RH_\infty^{k\times m}$ show that $z \mapsto \|G(z)\|$ is subharmonic.

d) For $G \in RH_\infty^{k\times m}$ prove the following "maximum modulus theorem":

$$\|G\|_\infty = \sup_{z\in\mathbb{C}^0\cup\mathbb{C}^+} \|G(z)\|.$$

9) Let L, W and Δ be transfer functions with the following properties:

- W and $T = L(1+L)^{-1}$ are stable and satisfy $\|WT\|_\infty \leq 1$.
- Δ has no poles in $\mathbb{C}^0$ and satisfies $|\Delta(i\omega)| < 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Prove the following facts:

a) $1 + TW\Delta$ and $(1+TW\Delta)^{-1}$ have no poles in $\mathbb{C}^0 \cup \{\infty\}$.

b) L and $L_\Delta := L(1+W\Delta)$ have identical poles in $\mathbb{C}^0$ (including multiplicities).

c) Suppose that $(1+L_\Delta)^{-1}$ is stable. Then L and L_Δ have the same number of poles in $\mathbb{C}^+$.

Hint: To prove the third statement one can use Rouché's theorem.

10) With the transfer function $L_0 = N/D$ (where N, D are real coprime polynomials) and $T > 0$ define the loop transfer function $L_T(s) = L_0(s)e^{-sT}$.

a) Show that the poles of $(1+L_T)^{-1}$ in $\mathbb{C}_0 \cup \mathbb{C}^+$ are given by the zeros of $p_T(s) := D(s) + N(s)e^{-sT}$ in $\mathbb{C}_0 \cup \mathbb{C}^+$.

b) If $|L_0(\infty)| < 1$ show that p_T has only finitely many zeros in $\mathbb{C}_0 \cup \mathbb{C}^+$.

c) If $|L_0(\infty)| < 1$ and p_T has no zeros in $\mathbb{C}_0 \cup \mathbb{C}^+$ show that $(1+L_T)^{-1}$ is analytic and bounded in $\mathbb{C}^+$.

d) Given constants $M > 0$, $R > 0$, show that there exists $T_0 > 0$ such that $M|1 - e^{-i\omega T}| < 1$ for all $\omega \in [-R, R]$, $T \in [0, T_0]$.

e) Suppose that $|L_0(\infty)| < 1$ and that p_0 has all its zeros in $\mathbb{C}^-$. Show that there exists some $T_0 > 0$ such that p_T has only zeros in $\mathbb{C}^-$ for $T \in [0, T_0)$. You can assume that L_0 has no poles in $\mathbb{C}^0$.
Hint: Use the Nyquist criterion for L_T.

f) If $|L_0(\infty)| > 1$ show that for all $T > 0$ the polynomial p_T has at least one zero in $\mathbb{C}^0 \cup \mathbb{C}^+$.
Hint: Consider the zeros of the function $f(s) = e^{T/s}L_0(s^{-1})^{-1} + 1$ with an essential singularity at 0 and apply Picard's Great Theorem without proof.

3 Stabilizing Controllers for Interconnections

In the previous section we have discussed some robustness issues for SISO systems. In practice, however, one encounters *interconnections* of multi input multi output (MIMO) systems. In the most simplest case, typical components of such an interconnection are a model of a considered physical plant and a to-be-designed feedback controller.

Before we deal with robustness for such interconnections of systems we consider the precise definition of when a system is internally stabilized by a controller.

3.1 A Specific Tracking Interconnection

To be concrete, let us look at the typical one-degree of freedom control configuration in Figure 17. Here G is the plant model, K is the to-be-designed controller, r is the reference input signal, d is a disturbance at the plant's output, n is measurement noise, e is the tracking error, u is the control input, and y is the measured output.

It is important to note that we have explicitly specified those signals that are of interest to us:

- Signals that affect the interconnection and cannot be influenced by the controller: r, d, n.
- Signals with which we characterize whether the controller achieves the desired goal: e should be kept as small as possible for all inputs r, d, n in a certain class.
- Signals via which the plant can be controlled: u.
- Signals that are available for control: y.

The interconnection does not only comprise the system components (G, K) and how the signals that are processed by these components are related to each other, but it also specifies those signals (e and r, d, n) that are related to the targeted task of the controller.

The corresponding *open-loop interconnection* is simply obtained by disconnecting the controller as shown in Figure 18.

It is straightforward to arrive, without any computation, at the following input-output description of the open-loop interconnection:

$$\left(\begin{array}{c} e \\ \hline y \end{array}\right) = \left(\begin{array}{ccc|c} I & 0 & -I & G \\ \hline -I & -I & I & -G \end{array}\right) \left(\begin{array}{c} d \\ n \\ r \\ \hline u \end{array}\right).$$

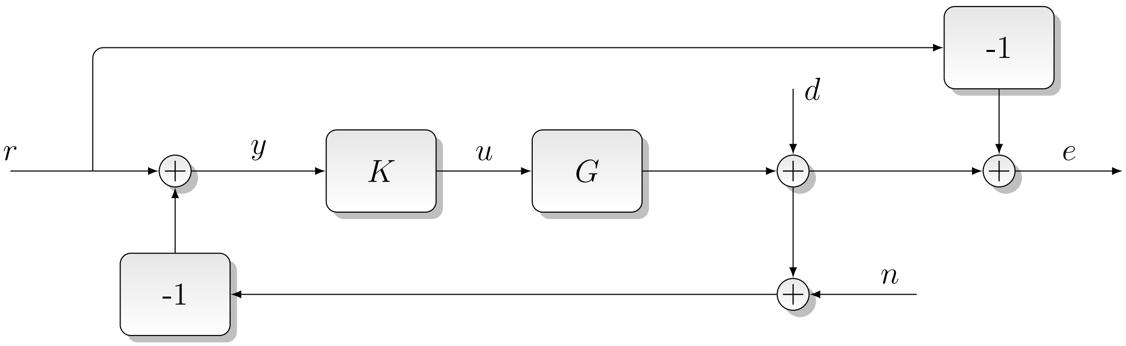


Figure 17: Closed-loop interconnection

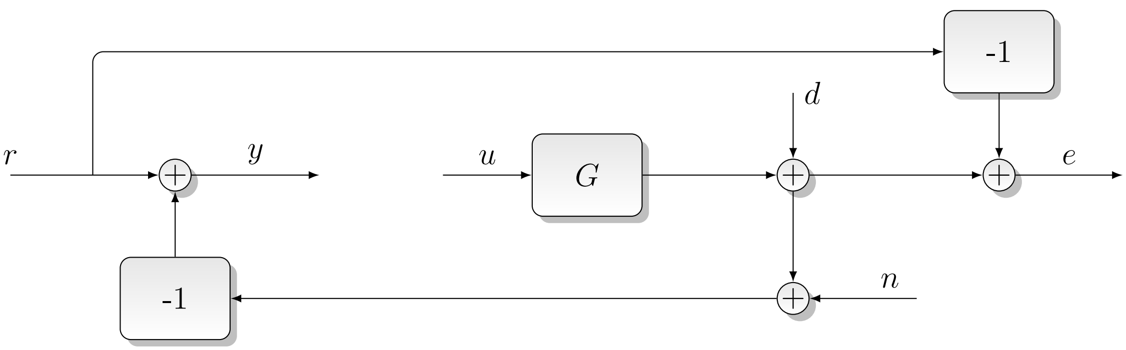


Figure 18: Open-loop interconnection that corresponds to Figure 17

The input-output description of the closed-loop interconnection is then obtained by closing the loop as

$$u = Ky.$$

A simple calculation reveals that

$$e = \left[\begin{pmatrix} I & 0 & -I \end{pmatrix} + GK(I - (-G)K)^{-1} \begin{pmatrix} -I & -I & I \end{pmatrix} \right] \begin{pmatrix} d \\ n \\ r \end{pmatrix}$$

what can be simplified to

$$e = \begin{pmatrix} (I + GK)^{-1} & -GK(I + GK)^{-1} & -(I + GK)^{-1} \end{pmatrix} \begin{pmatrix} d \\ n \\ r \end{pmatrix}.$$

As expected for this specific interconnection, we arrive at

$$e = \begin{pmatrix} S & -T & -S \end{pmatrix} \begin{pmatrix} d \\ n \\ r \end{pmatrix}$$

with sensitivity $S = (I + GK)^{-1}$ and complementary sensitivity $T = GK(I + GK)^{-1}$.

Let us now extract a general scheme from this specific example.

3.2 The General Framework

In an arbitrary closed-loop interconnection structure, let

- w denote the signal that affects the system and cannot be influence by the controller. w is called **generalized disturbance**. (In our example, $w = \begin{pmatrix} d \\ n \\ r \end{pmatrix}$.)
- z denote the signal that allows to characterize whether a controller has certain desired properties. z is called **controlled variable**. (In our example, $z = e$.)
- u denote the output signal of the controller, the so-called **control input**. (In our example it's just u.)
- y denote the signal that enters the controller, the so-called **measurement output**. (In our example it's just y.)

Any open-loop interconnection can then be generally described by (Figure 19)

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \tag{3.1}$$

where the system P comprises the subsystems that are involved in the interconnection and the manner how these subsystems are connected with each other.

Even if we start with an interconnection of SISO systems, the resulting open-loop interconnection will generally be described by a MIMO system since one has to stack several signals with only one component to vector valued signals.

In these whole notes we start from the fundamental hypothesis that P is an LTI system. We denote the corresponding transfer matrix with the same symbol as

$$P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix}.$$

Let

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \tag{3.2}$$

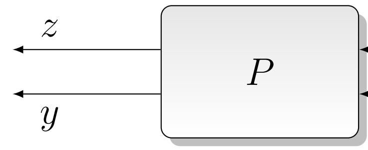


Figure 19: General open-loop interconnection

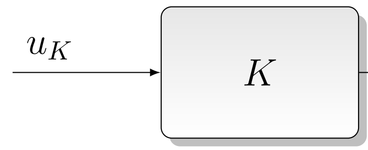


Figure 20: Controller

denote a stabilizable and detectable state-space realization of P.

A controller (Figure 20) is any LTI system

$$y_K = K u_K. \tag{3.3}$$

It can be described in the frequency domain by specifying its transfer matrix

$$K(s)$$

or via the stabilizable and detectable state-space realization

$$\begin{aligned} \dot{x}_K &= A_K x_K + B_K u_K \\ y_K &= C_K x_K + D_K u_K. \end{aligned} \tag{3.4}$$

The interconnection of the controller and the open-loop system as

$$u_K = y \quad \text{and} \quad u = y_K$$

leads to the closed-loop interconnection as depicted in Figure 21.

Remark 3.1 To have a minimal dimensions of the matrices and, hence, reduce the effort for all subsequent computations, one should rather work with *minimal* (controllable and observable) realizations for P and K. One can take these stronger hypothesis as the basis for the discussion throughout these notes without the need for any modification.

3.3 Stabilizing Controllers - State-Space Descriptions

Let us now first compute a realization of the interconnection as

$$\begin{pmatrix} u \\ u_K \end{pmatrix} = \begin{pmatrix} y_K \\ y \end{pmatrix} \tag{3.5}$$

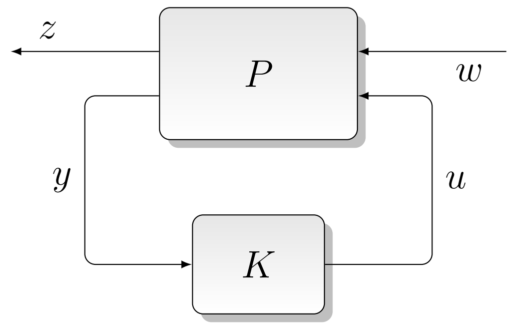


Figure 21: General closed-loop interconnection

of the system (3.2) and the controller (3.4).

For that purpose it is advantageous to merge the descriptions of (3.2) and (3.4) as

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \\ \hline y_K \\ y \end{pmatrix} = \left(\begin{array}{cc|c|cc} A & 0 & B_1 & B_2 & 0 \\ 0 & A_K & 0 & 0 & B_K \\ \hline C_1 & 0 & D_{11} & D_{12} & 0 \\ \hline 0 & C_K & 0 & 0 & D_K \\ C_2 & 0 & D_{21} & D_{22} & 0 \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline w \\ \hline u \\ u_K \end{pmatrix}.$$

To simplify the calculations notationally, let us introduce the abbreviation

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ \boldsymbol{C}_2 & \boldsymbol{D}_{21} & \boldsymbol{D}_{22} \end{pmatrix} := \left(\begin{array}{cc|c|cc} A & 0 & B_1 & B_2 & 0 \\ 0 & A_K & 0 & 0 & B_K \\ \hline C_1 & 0 & D_{11} & D_{12} & 0 \\ \hline 0 & C_K & 0 & 0 & D_K \\ C_2 & 0 & D_{21} & D_{22} & 0 \end{array}\right). \tag{3.6}$$

The interconnection (3.5) leads to

$$\begin{pmatrix} y_K \\ y \end{pmatrix} = \left(\boldsymbol{C}_2 \middle| \boldsymbol{D}_{21} \right) \begin{pmatrix} x \\ x_K \\ \hline w \end{pmatrix} + \boldsymbol{D}_{22} \begin{pmatrix} y_K \\ y \end{pmatrix}$$

or

$$[I - \boldsymbol{D}_{22}] \begin{pmatrix} y_K \\ y \end{pmatrix} = \left(\boldsymbol{C}_2 \middle| \boldsymbol{D}_{21} \right) \begin{pmatrix} x \\ x_K \\ \hline w \end{pmatrix}.$$

If $I - \boldsymbol{D}_{22}$ is non-singular, we arrive at

$$\begin{pmatrix} y_K \\ y \end{pmatrix} = [I - \boldsymbol{D}_{22}]^{-1} \left(\boldsymbol{C}_2 \middle| \boldsymbol{D}_{21} \right) \begin{pmatrix} x \\ x_K \\ \hline w \end{pmatrix}$$

what finally leads to

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \end{pmatrix} = \left(\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}_2 \\ \boldsymbol{D}_{12} \end{pmatrix} [I - \boldsymbol{D}_{22}]^{-1} \begin{pmatrix} \boldsymbol{C}_2 & \boldsymbol{D}_{21} \end{pmatrix} \right) \begin{pmatrix} x \\ x_K \\ \hline w \end{pmatrix}$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{A} + \boldsymbol{B}_2[I - \boldsymbol{D}_{22}]^{-1}\boldsymbol{C}_2 & \boldsymbol{B}_1 + \boldsymbol{B}_2[I - \boldsymbol{D}_{22}]^{-1}\boldsymbol{D}_{21} \\ \hline \boldsymbol{C}_1 + \boldsymbol{D}_{12}[I - \boldsymbol{D}_{22}]^{-1}\boldsymbol{C}_2 & \boldsymbol{D}_1 + \boldsymbol{D}_{12}[I - \boldsymbol{D}_{22}]^{-1}\boldsymbol{D}_{21} \end{array} \right) \begin{pmatrix} x \\ x_K \\ \hline w \end{pmatrix}. \tag{3.7}$$

This is an explicit formula for a state-space representation of the closed-loop interconnection.

On our way to derive this formula we assumed that $I - \boldsymbol{D}_{22}$ is non-singular. This is a condition to ensure that we could indeed close the loop; this is the reason why it is often called a well-posedness condition for the interconnection.

Any controller should at least be chosen such that the interconnection is well-posed. In addition, we require that the controller stabilizes the interconnection. This will just amount to requiring that the matrix $\boldsymbol{A} + \boldsymbol{B}_2[I - \boldsymbol{D}_{22}]^{-1}\boldsymbol{C}_2$ which defines the dynamics of the interconnection is stable.

We arrive at the following fundamental definition of when the controller (3.4) stabilizes the open-loop system (3.2).

Definition 3.2 *The controller (3.4)* **stabilizes** *the system (3.2) if*

$$\begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix} \text{ is non-singular} \tag{3.8}$$

and if

$$\begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & C_K \\ C_2 & 0 \end{pmatrix} \tag{3.9}$$

has all its eigenvalues in the open left-half plane $\mathbb{C}_<$.

Remark 3.3

- Verifying whether K stabilizes P is very simple: First check whether the realizations of both P and K are stabilizable and detectable, then check (3.8), and finally verify whether (3.9) is stable.

- If the realizations of both P and K are stabilizable and detectable, the realization of the closed-loop interconnection (3.7) is in general not stabilizable and not detectable.

- Note that the definition only involves the matrices
$$\begin{pmatrix} A & B_2 \\ C_2 & D_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}.$$
The matrices B_1 and C_1 only play a role in requiring that $\left(A, \begin{pmatrix} B_1 & B_2 \end{pmatrix}\right)$ is stabilizable and $\left(A, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\right)$ is detectable.

- The same definition is in effect if the channel $w \to z$ is void and the system (3.2) just reads as
$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A & B_2 \\ C_2 & D_{22} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Note that the formulas (3.7) for the closed-loop interconnection simplify considerably if either D_{22} or D_K vanish. Let us look at the case when
$$D_{22} = 0.$$
Then (3.8) is always true. Due to
$$\begin{pmatrix} I & -D_K \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & D_K \\ 0 & I \end{pmatrix},$$
a straightforward calculation reveals that (3.7) now reads as
$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \end{pmatrix} = \left(\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ C_2 B_K & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline w \end{pmatrix}.$$
Then the matrix (3.9) just equals
$$\begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ C_2 B_K & A_K \end{pmatrix}.$$

Example 3.4 Consider again the classical configuration in section 3.1 with

$$G(s) = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2} = \left[\begin{array}{ccc|c} -0.1 & 1 & 0 & 0 \\ 0 & -20 & 1 & 0 \\ 0 & 0 & -20 & 64 \\ \hline 125 & 0 & 0 & 0 \end{array}\right](s).$$

and a controller

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = \left[\begin{array}{c|c} -3.3 & 1.3 \\ \hline 1.7 & 0.3 \end{array}\right]$$

with stabilizable and detectable realization. A stabilizable and detectable realization for the open-loop interconnection P (depicted in Figure 18) is given by

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{ccc|ccc|c} -0.1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 & 0 & 64 \\ \hline 125 & 0 & 0 & 1 & 0 & -1 & 0 \\ \hline -125 & 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right].$$

Since $D_{22} = 0$ clearly implies (3.8) and since the eigenvalues of (3.9) are contained in $\mathbb{C}^-$, we can conclude that K stabilizes P.

Next we want to derive a characterization of stabilizing controllers in terms of transfer matrices. To do so, we first introduce linear fractional transformations.

3.4 Linear Fractional Transformations

Suppose P and K are given transfer matrices. Then the so-called **lower linear fractional transformation** $P \star K$ of P and K is defined as follows: Partition

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

such that $P_{22}K$ is *square*, check whether the rational matrix $I - P_{22}K$ has a rational inverse, and set

$$P \star K := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

In the literature, the expression $\mathcal{F}_l(P, K)$ is often used instead of $P \star K$. Since a lower linear fractional transformation is a particular form of a more general operation that carries the name star-product, we prefer the symbol $P \star K$.

Similarly, the **upper linear fractional transformation** $\Delta \star P$ of the rational matrices Δ and P is defined as follows: Choose a partition

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

such that $P_{11}\Delta$ is square, check whether the rational matrix $I - P_{11}\Delta$ has a rational inverse, and set

$$\Delta \star P := P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}.$$

One often finds the notation $\mathcal{F}_u(P, \Delta)$ in the literature where one has to note that, unfortunately, the matrices Δ and P appear in reverse order.

At this point, $P \star K$ and $\Delta \star P$ should be just viewed as abbreviations for the formulas given above. The discussion to follow will reveal their system theoretic relevance.

3.5 Stabilizing Controllers - Input-Output Descriptions

Let us first see how to determine an input-ouput description of the closed-loop interconnection as in Figure 21. For that purpose we only need to eliminate the signals u, y in

$$z = P_{11}w + P_{12}u, \quad y = P_{21}w + P_{22}u, \quad u = Ky.$$

The last two relations lead to $y = P_{21}w + P_{22}Ky$ or $(I - P_{22}K)y = P_{21}w$. If $I - P_{22}K$ does have a proper inverse, we obtain $y = (I - P_{22}K)^{-1}P_{21}$ and, finally,

$$z = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]w. \tag{3.10}$$

This is a general formula how to obtain, from the input-ouput description P and from that of the controller K, the corresponding input-output description of the closed-loop interconnection. If we recall the definitions in Section 3.4, we observe that the closed-loop input-output description by performing the lower linear fractional transformation of P with K which has been denoted as $P \star K$:

$$z = (P \star K)w.$$

This is the mere reason why these fractional transformations play such an important role in these notes and, in general, in robust control.

Note that (3.10) gives the transfer matrix that is defined by (3.7) and, conversely, (3.7) is a state-space realization of (3.10).

Let us now observe that $I - P_{22}K$ has a proper inverse if and only if $I - P_{22}(\infty)K(\infty)$ is non-singular. If we look back to the realizations (3.2) and (3.4), this just means that $I - D_{22}D_K$ is non-singular, what is in turn equivalent to (3.8).

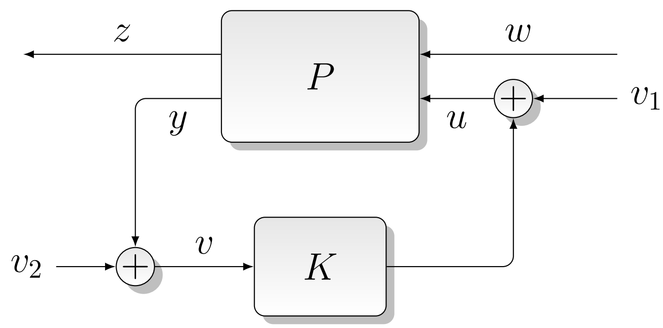


Figure 22: Interconnection to test whether K stabilizies P.

Unfortunately, the relation for stability is not as straightforward. In general, if (3.7) is stable, the transfer matrix defined through (3.10) is stable as well. However, the realization (3.7) is not necessarily stabilizable or detectable. Therefore, even if (3.10) defines a stable transfer matrix, the system (3.7) is not necessarily stable. For checking whether K stabilizes P, it hence *does not suffice* to simply verify whether $P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$ defines a stable transfer matrix.

This indicates that we have to check more transfer matrices in the loop in Figure 21 than just the one explicitly displayed by the channel $w \to z$ in order to guarantee that K stabilizes P. It turns out that Figure 22 gives a suitable setup to define all the relevant transfer matrices that have to be tested.

Theorem 3.5 *K stabilizes P if and only if the interconnection as depicted in Figure 22 and defined through the relations*

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix}, \quad u = Kv + v_1, \quad v = y + v_2$$

or, equivalently, by

$$\left(\begin{array}{c} z \\ \hline v_1 \\ v_2 \end{array}\right) = \left(\begin{array}{c|cc} P_{11} & P_{12} & 0 \\ \hline 0 & I & -K \\ -P_{21} & -P_{22} & I \end{array}\right) \left(\begin{array}{c} w \\ \hline u \\ v \end{array}\right) \tag{3.11}$$

defines a proper transfer matrix

$$\begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} z \\ u \\ v \end{pmatrix} \tag{3.12}$$

that is stable.

With this result we can test directly on the basis of the transfer matrices whether K stabilizes P: One has to check whether the relations (3.11) define a proper and stable transfer matrix (3.12).

Let us first clarify what this means exactly. Clearly, (3.11) can be rewritten as

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -P_{21} \end{pmatrix} w + \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad z = P_{11}w + \begin{pmatrix} P_{12} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.13}$$

These relations define a proper transfer matrix (3.12) if and only if

$$\begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix} \text{ has a proper inverse.}$$

As well-known, this is true if and only if

$$\begin{pmatrix} I & -K(\infty) \\ -P_{22}(\infty) & I \end{pmatrix} \text{ is non-singular.}$$

Indeed, under this hypothesis, the first relation in (3.13) is equivalent to

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ P_{21} \end{pmatrix} w + \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Hence (3.13) is nothing but

$$\left(\begin{array}{c} z \\ \hline u \\ v \end{array}\right) = \left(\begin{array}{c|c} P_{11} + \begin{pmatrix} P_{12} & 0 \end{pmatrix} \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ P_{21} \end{pmatrix} & \begin{pmatrix} P_{12} & 0 \end{pmatrix} \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \\ \hline \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ P_{21} \end{pmatrix} & \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \end{array}\right) \left(\begin{array}{c} w \\ \hline v_1 \\ v_2 \end{array}\right).$$

If we recall the formula

$$\begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} = \begin{pmatrix} (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{pmatrix},$$

this can be rewritten to

$$\left(\begin{array}{c} z \\ \hline u \\ v \end{array}\right) = \left(\begin{array}{c|cc} P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} & P_{12}(I - KP_{22})^{-1} & P_{12}K(I - P_{22}K)^{-1} \\ \hline K(I - P_{22}K)^{-1}P_{21} & (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{21} & (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{array}\right) \left(\begin{array}{c} w \\ \hline v_1 \\ v_2 \end{array}\right). \tag{3.14}$$

We have arrived at a more explicit reformulation of Theorem 3.5.

Corollary 3.6 *K stabilizes P if and only if $I - P_{22}K$ has a proper inverse and all nine transfer matrices in (3.14) are stable.*

Remark 3.7 If the channel $w \to z$ is absent, the characterizing conditions in Theorem 3.5 or Corollary 3.6 read as follows: The transfer matrix

$$\begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}$$

has a proper and stable inverse. Since K and P_{22} are, in general, not stable, it does not suffice to simply verify whether the determinant of this matrix is stable; Lemma 1.4 does not apply!

Proof of Theorem 3.5. We have already clarified that (3.11) defines a proper transfer matrix (3.12) if and only if (3.8) is true. Let us hence assume the validity of (3.8).

Then we observe that (3.11) admits the state-space realization

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \\ \hline v_1 \\ v_2 \end{pmatrix} = \left(\begin{array}{cc|c|cc} A & 0 & B_1 & B_2 & 0 \\ 0 & A_K & 0 & 0 & B_K \\ \hline C_1 & 0 & D_{11} & D_{12} & 0 \\ \hline 0 & -C_K & 0 & I & -D_K \\ -C_2 & 0 & -D_{21} & -D_{22} & I \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline w \\ \hline u \\ v \end{pmatrix}. \tag{3.15}$$

Using the abbreviation (3.6), this is nothing but

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \\ \hline v_1 \\ v_2 \end{pmatrix} = \left(\begin{array}{c|c|c} \boldsymbol{A} & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \hline \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ \hline -\boldsymbol{C}_2 & -\boldsymbol{D}_{21} & I - \boldsymbol{D}_{22} \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline w \\ \hline u \\ v \end{pmatrix}.$$

By (3.8), $\tilde{\boldsymbol{D}}_{22} := I - \boldsymbol{D}_{22}$ is non-singular. The same calculation as performed earlier leads to a state-space realization of (3.12):

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline z \\ \hline u \\ v \end{pmatrix} = \left(\begin{array}{c|c|c} \boldsymbol{A} + \boldsymbol{B}_2\tilde{\boldsymbol{D}}_{22}^{-1}\boldsymbol{C}_2 & \boldsymbol{B}_1 + \boldsymbol{B}_2\tilde{\boldsymbol{D}}_{22}^{-1}\boldsymbol{D}_{21} & \boldsymbol{B}_2\tilde{\boldsymbol{D}}_{22}^{-1} \\ \hline \boldsymbol{C}_1 + \boldsymbol{D}_{12}\tilde{\boldsymbol{D}}_{22}^{-1}\boldsymbol{C}_2 & \boldsymbol{D}_1 + \boldsymbol{D}_{12}\tilde{\boldsymbol{D}}_{22}^{-1}\boldsymbol{D}_{21} & \boldsymbol{D}_{12}\tilde{\boldsymbol{D}}_{22}^{-1} \\ \hline \tilde{\boldsymbol{D}}_{22}^{-1}\boldsymbol{C}_2 & \tilde{\boldsymbol{D}}_{22}^{-1}\boldsymbol{D}_{21} & \tilde{\boldsymbol{D}}_{22}^{-1} \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline w \\ \hline v_1 \\ v_2 \end{pmatrix}. \tag{3.16}$$

Here is the crux of the proof: Since (3.2) and (3.4) are stabilizable/detectable realizations, one can easily verify with the Hautus test that (3.15) has the same property. This implies that the realization (3.16) is stabilizable and detectable as well.

Therefore we can conclude: The transfer matrix of (3.16) is stable *if and only if* the system (3.16) is stable *if and only if* $\boldsymbol{A} + \boldsymbol{B}_2\tilde{\boldsymbol{D}}_{22}^{-1}\boldsymbol{C}_2 = \boldsymbol{A} + \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2$ has all its eigenvalues in $\mathbb{C}_<$. Since we have guaranteed the validity of (3.8), this property is (by Definition 3.2) nothing but the fact that K stabilizes P.

■

3.6 Generalized Plants

Contrary to what one might expect, it is not possible to find a stabilizing controller K for any P.

Example 3.8 Let us consider (3.1) with

$$P(s) = \begin{pmatrix} 1 & 1/s \\ 1 & 1/(s+1) \end{pmatrix}.$$

We claim that there is no $K(s)$ that stabilizes $P(s)$. Reason: Suppose we found a $K(s)$ that stabilizes $P(s)$. Then the two transfer functions

$$P_{12}(s)(I - K(s)P_{22}(s))^{-1} = \frac{1}{s}\frac{1}{1 - \frac{K(s)}{s+1}}$$

$$P_{12}(s)(I - K(s)P_{22}(s))^{-1}K(s) = \frac{K(s)}{s - \frac{s}{s+1}K(s)} = \frac{1}{s}\frac{1}{\frac{1}{K(s)} - \frac{1}{s+1}}$$

are stable. But this cannot be true. To show that, we distinguish two cases:

- Suppose $K(s)$ has no pole in 0. Then the denominator of $\frac{1}{1-\frac{K(s)}{s+1}}$ is finite in $s = 0$ such that this function cannot have a zero in $s = 0$. This implies that the first of the above two transfer functions has a pole in 0, i.e., it is unstable.

- Suppose $K(s)$ does have a pole in 0. Then $\frac{1}{K(s)}$ vanishes in $s = 0$ such that $\frac{1}{\frac{1}{K(s)} - \frac{1}{s+1}}$ takes the value -1 in $s = 0$. Hence, the second of the above two transfer functions has a pole in 0 and is, thus, unstable.

We arrive at the contradiction that at least one of the above two transfer functions is always unstable. Roughly speaking, the pole $s = 0$ of the transfer function $P_{12}(s) = \frac{1}{s}$ cannot be stabilized via feeding y back to u since this is not a pole of $P_{22}(s)$ as well.

Our theory will be based on the hypothesis that P does in fact admit a stabilizing controller. For such open-loop interconnections we introduce a particular name.

Definition 3.9 *If there exists at least one controller K that stabilizes the open-loop interconnection P, we call P a* **generalized plant**.

Fortunately, one can very easily check whether a given P is a generalized plant or not. We first formulate a test for the state-space description of P.

Theorem 3.10 *P with the stabilizable/detectable realization (3.2) is a generalized plant if and only if (A, B_2) is stabilizable and (A, C_2) is detectable.*

Since the realization (3.2) is stabilizable, we know that $\left(A, \begin{pmatrix} B_1 & B_2 \end{pmatrix}\right)$ is stabilizable. This does clearly not imply, in general, that the pair (A, B_2) defining a system with fewer inputs is stabilizable. A similar remark holds for detectability.

Let us now assume that (A, B_2) is stabilizable and (A, C_2) is detectable. Then we can explicitly construct a controller that stabilizes P. In fact, stabilizability of (A, B_2) and detectability of (A, C_2) imply that there exist F and L such that $A + B_2F$ and $A + LC_2$ are stable. Let us now take the controller K that is defined through

$$\dot{x}_K = (A + B_2F + LC_2 + LD_{22}F)x_K - Ly, \quad u = Fx_K.$$

Note that this is nothing but the standard observer-based controller which one would design for the system

$$\dot{x} = Ax + B_2u, \quad y = C_2x + D_{22}u.$$

It is simple to check that K indeed stabilizes P. First, K is strictly proper ($D_K = 0$) such that (3.8) is obviously true. Second, let us look at

$$\begin{aligned}
&\begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} I & 0 \\ -D_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & C_K \\ C_2 & 0 \end{pmatrix} = \\
&\quad = \begin{pmatrix} A & 0 \\ 0 & A + B_2F + LC_2 + LD_{22}F \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & -L \end{pmatrix} \begin{pmatrix} I & 0 \\ D_{22} & I \end{pmatrix} \begin{pmatrix} 0 & F \\ C_2 & 0 \end{pmatrix} = \\
&\quad\quad = \begin{pmatrix} A & 0 \\ 0 & A + B_2F + LC_2 + LD_{22}F \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & -L \end{pmatrix} \begin{pmatrix} 0 & F \\ C_2 & D_{22}F \end{pmatrix} = \\
&\quad\quad\quad = \begin{pmatrix} A & B_2F \\ -LC_2 & A + B_2F + LC_2 \end{pmatrix}.
\end{aligned}$$

We claim that this matrix is stable. This should be known from classical theory. However, it can be verified by performing the similarity transformation (error dynamics!)

$$\begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} A & B_2F \\ -LC_2 & A+B_2F+LC_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}^{-1}$$

to arrive at

$$\begin{pmatrix} A+B_2F & B_2F \\ 0 & A+LC_2 \end{pmatrix}$$

which is, obviously, stable since the diagonal blocks are.

This was the proof of the if-part in Theorem 3.10 with an explicit construction of a stabilizing controller.

Proof of only if. To finish the proof, we have to show: If there exists a K that stabilizes P, then (A, B_2) is stabilizable and (A, C_2) is detectable. If K stabilizes P, we know by definition that

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & C_K \\ C_2 & 0 \end{pmatrix}$$

is stable. This implies that (A, C_2) is detectable. Let us prove this fact with the Hautus test: Suppose $Ax = \lambda x$, $x \neq 0$, and $C_2x = 0$. Then we observe that

$$\mathcal{A} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Hence $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an eigenvector of $\mathcal{A}$ with eigenvalue λ. Since $\mathcal{A}$ is stable, we infer $\mathrm{Re}(\lambda) < 0$. This proves that (A, C_2) is detectable.

Task. Show in a similar fashion that (A, B_2) is stabilizable what finishes the proof.

■

Remark 3.11 If the channel $w \to z$ is absent, then (A, B_2) and (A, C_2) are obviously stabilizable and detectable. (The matrices B_1, C_2, D_{11}, D_{12}, D_{21} in (3.2) are void.) Then there *always* exists a controller K that stabilizes P.

This last remark reveals that we can always find a $u = Ky$ that stabilizes $y = P_{22}u$. This leads us to a input-output test of whether P is a generalized plant or not.

Theorem 3.12 *Let $u = Ky$ be any controller that stabilizes $y = P_{22}u$. Then P is a generalized plant if and only if this controller K stabilizes the open-loop interconnection P.*

Again, this test is easy to perform: Find an (always existing) K that stabilizes P_{22}, and verify that this K renders all the nine transfer matrices in (3.14) stable. If yes, P is a generalized plant, if no, P is not.

Proof. Let K stabilize P_{22}.

If K also stabilizes P, we infer that there exists a stabilizing controller and, hence, P is a generalized plant.

Conversely, let P be a generalized plant. We intend to show that K not only stabilizes P_{22} but even P. We proceed with state-space arguments. Recall that $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ admits the state-space realization

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline v_1 \\ v_2 \end{pmatrix} = \left(\begin{array}{cc|cc} A & 0 & B_2 & 0 \\ 0 & A_K & 0 & B_K \\ \hline 0 & -C_K & I & -D_K \\ -C_2 & 0 & -D_{22} & I \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline u \\ v \end{pmatrix}. \tag{3.17}$$

Using the abbreviation (3.6), this is nothing but

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline v_1 \\ v_2 \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B}_2 \\ \hline -\boldsymbol{C}_2 & I - \boldsymbol{D}_{22} \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline u \\ v \end{pmatrix}.$$

By (3.8), $I - \boldsymbol{D}_{22}$ is non-singular. Again, the same calculation as earlier leads to a state-space realization of $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ given by

$$\begin{pmatrix} \dot{x} \\ \dot{x}_K \\ \hline u \\ v \end{pmatrix} = \left(\begin{array}{c|c} \boldsymbol{A} + \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2 & \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1} \\ \hline (I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2 & (I - \boldsymbol{D}_{22})^{-1} \end{array}\right) \begin{pmatrix} x \\ x_K \\ \hline v_1 \\ v_2 \end{pmatrix}. \tag{3.18}$$

Since P is a generalized plant, (A, B_2) is stabilizable and (A, C_2) is detectable. Therefore, the same is true for (3.17) and, similarly as in the proof of Theorem 3.5, also for (3.18). Since K stabilizes P_{22}, the transfer matrix defined through (3.18) is stable. Since this realization is stabilizable and detectable, we can conclude that $\boldsymbol{A} + \boldsymbol{B}_2(I - \boldsymbol{D}_{22})^{-1}\boldsymbol{C}_2$ is actually stable. Hence K stabilizes also P by definition.

■

3.7 Summary

For a specific control task, extract the open-loop interconnection (3.1).

Then test whether this open-loop interconnection defines a generalized plant by applying either one of the following procedures:

- Find a state-space realization (3.2) of P for which $\left(A, \begin{pmatrix} B_1 & B_2 \end{pmatrix}\right)$ is stabilizable (or even controllable) and $\left(A, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\right)$ is detectable (or even observable), and check whether (A, B_2) is stabilizable and (A, C_2) is detectable. If yes, P is a generalized plant, if no, P is not.
- Find any K such that $\begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}$ does have a proper and stable inverse. Then verify whether this K renders all transfer matrices in (3.14) stable. If yes, P is a generalized plant, if no, P is not.

If P turns out to be no generalized plant, the interconnection under consideration is not suitable for the theory to be developed in these notes.

Suppose K stabilizes P. Then the closed-loop interconnection is described as

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w = (P \star K)w$$

In the state-space, the closed-loop system admits the realization (3.7) with the abbreviations (3.6).

3.8 Back to the Tracking Interconnection

Let us come back to the specific tracking interconnection in Figure 17 for which we have obtained

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \left(\begin{array}{ccc|c} I & 0 & -I & G \\ \hline -I & -I & I & -G \end{array}\right).$$

We claim that this is a generalized plant.

Input-output test: Let K stabilize $P_{22} = -G$. This means that that

$$\begin{pmatrix} (I - KP_{22})^{-1} & K(I - P_{22}K)^{-1} \\ (I - P_{22}K)^{-1}P_{22} & (I - P_{22}K)^{-1} \end{pmatrix} = \begin{pmatrix} (I + KG)^{-1} & K(I + GK)^{-1} \\ -(I + GK)^{-1}G & (I + GK)^{-1} \end{pmatrix}$$

is well-defined and stable. Let us now look at (3.14). Since $P_{21}(s) = \begin{pmatrix} -I & -I & I \end{pmatrix}$ is stable, the same is true of $K(I - P_{22}K)^{-1}P_{21}$ and $(I - P_{22}K)^{-1}P_{21}$. Since $P_{12} = G$, we infer

$$P_{12}(I - KP_{22})^{-1} = G(I + KG)^{-1} = (I + GK)^{-1}G$$

and

$$P_{12}K(I - KP_{22})^{-1} = KG(I + KG)^{-1} = I - (I + KG)^{-1}$$

that are both stable. Hence it remains to check stability of

$$P \star K = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} :$$

We have just seen that $P_{12}K(I - KP_{22})^{-1}$ is stable. Since the same is true for $P_{11} = \begin{pmatrix} I & 0 & -I \end{pmatrix}$ and P_{21}, we can indeed conclude that $P \star K$ is stable. This reveals that all nine transfer matrices in (3.14) are stable. By Theorem 3.12, P is a generalized plant.

State-space test: Let us assume that

$$G(s) = C_G(sI - A_G)^{-1}B_G + D_G$$

is a minimal realization. Then we observe that

$$P(s) = \left(\begin{array}{c} C_G \\ \hline -C_G \end{array}\right)(sI - A_G)^{-1}\left(\begin{array}{ccc|c} 0 & 0 & 0 & B_G \end{array}\right) + \left(\begin{array}{ccc|c} I & 0 & -I & D_G \\ \hline -I & -I & I & -D_G \end{array}\right)$$

and hence $P(s)$ admits a minimal realization with the matrix

$$\begin{pmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{pmatrix} = \left(\begin{array}{c|ccc|c} A_G & 0 & 0 & 0 & B_G \\ \hline C_G & I & 0 & -I & D_G \\ \hline -C_G & -I & -I & I & -D_G \end{array}\right).$$

Since $(A, B_2) = (A_G, B_G)$ is controllable and $(A, C_2) = (A_G, C_G)$ is observable, Theorem 3.10 implies that P is a generalized plant.

Note that all these tests are very simple, mainly due to the simplicity of the feedback interconnection in Figure 17 under scrutiny. In practical circumstances one might encounter a much more complicated configuration where the tests have to be performed numerically. We recommend Matlab's Robust Control Toolbox command `sysic` to easily build state-space realization even for complicated interconnections.

Exercises

1) Let P be a stable stable LTI system.

   a) Show that K stabilizes P if and only if $I - P_{22}K$ has a proper inverse and $K(I - P_{22}K)^{-1}$ is stable. (It suffices to check *one* instead of *nine* transfer matrices.)

      Is the same statement true if we replace $K(I - P_{22}K)^{-1}$ with $(I - P_{22}K)^{-1}$?

   b) Show that the set of closed-loop transfer matrices $P \star K$ where K varies over all controllers that stabilize P is given by the set of all

   $$P_{11} + P_{12}QP_{21}$$

   where Q is a free parameter in RH_∞. What is the relation between K and Q? (This is the so-called Youla parameterization. Note that K enters in $P \star K$ in a non-linear fashion, whereas Q enters $P_{11} + P_{12}QP_{21}$ in an affine fashion. Hence the change of parameters $K \to Q$ leads to an affine dependence of the closed-loop system on the so-called Youla-parameter. All this can be extended to general systems P that are not necessarily stable.)

2) Which of the following transfer matrices $P(s)$ define a generalized plant:

$$\begin{pmatrix} 1/(s+1) & 1/(s+2) \\ 1/(s+3) & 1/s \end{pmatrix}, \quad \begin{pmatrix} 1/(s^2+s) & 1/(s+2) \\ 1/(s+3) & 1/s \end{pmatrix}$$
$$\begin{pmatrix} 1/(s+1) & 1/(s^2+2s) \\ 1/(s+3) & 1/s \end{pmatrix}, \quad \begin{pmatrix} 1/(s+1) & 1/(s+2) \\ 1/(s+3) & 1/s^2 \end{pmatrix}?$$

It is assumed that all signals in $\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix}$ have one component.

3) Suppose you have given the interconnection in Figure 23. We view G_j, $j = 1,2,3,4,5,6,7$ as possibly MIMO system components, and K_j, $j = 1,2$, are possibly MIMO controller blocks.

   a) Compute the description P of the open-loop interconnection in terms of G_j. Mind the fact that all components can have multiple inputs and outputs.

   b) Find two examples with simple SISO components G_j such that the resulting two open-loop interconnections P_1, P_2 have the following properties:

      P_1 is no generalized plant. There exists a controller K that renders $S(P_2, K)$ stable but that does not stabilize P_2.

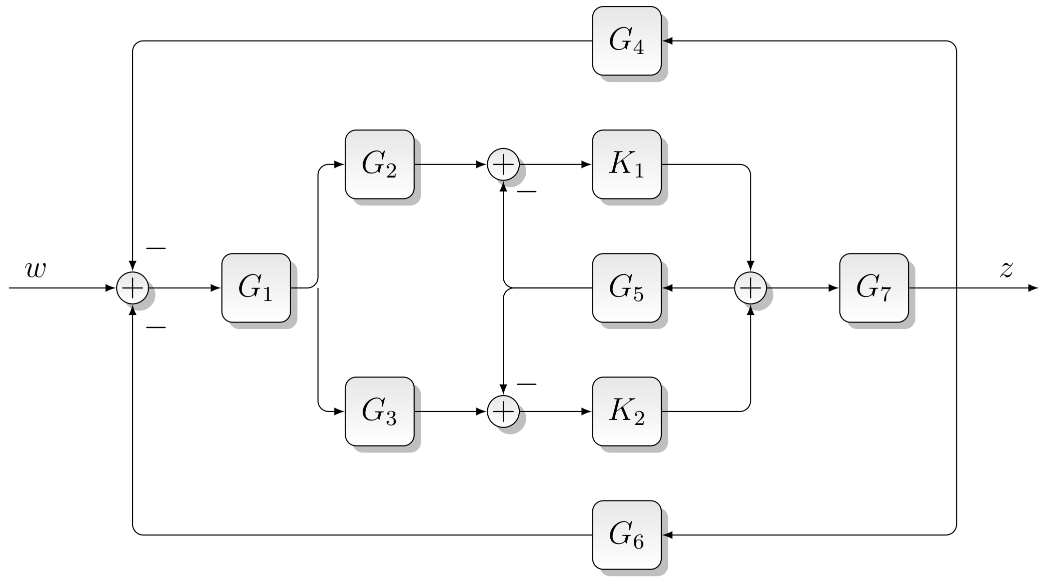


Figure 23: An interconnection

Is it possible to take $P_1 = P_2$?

c) (Matlab) Choose

$$G_1(s) = 1,\ G_2(s) = \frac{1}{s-1},\ G_3(s) = \frac{s+1}{s^2+1},\ G_4(s) = 0,$$
$$G_5(s) = \frac{1}{s},\ G_6(s) = 1,\ G_7(s) = \frac{s+2}{(s+3)(s-2)}.$$

Show that P is a generalized plant. Design a controller K that stabilizes P. Explain how you obtain K, and how you check whether K indeed stabilizes P. Draw a Bode magnitude plot of the resulting closed loop system $P \star K$.

4 Robust Stability Analysis

All mathematical models of a physical system suffer from inaccuracies that result from non-exact measurements or from the general inability to capture all phenomena that are involved in the dynamics of the considered system. Even if it is possible to accurately model a system, the resulting descriptions are often too complex to allow for a subsequent analysis, not to speak of the design of a controller. Hence one rather chooses for a simple model and takes a certain error between the simplified and the more complex model into account.

Therefore, there is always a mismatch between the model and the system to be investigated. A control engineer calls this mismatch *uncertainty*. Note that this is an abuse of notation since neither the system nor the model are uncertain; it is rather our knowledge about the actual physical system that we could call uncertain.

The main goal of robust control techniques is to take these uncertainties in a systematic fashion into account when analyzing a control system or when designing a controller for it.

In order to do so, one has to arrive at a mathematical description of the uncertainties. Sometimes it is pretty obvious what to call an uncertainty (such as parameter variations in a good physical model), but sometimes one just has to postulate a certain structure of the uncertainty. Instead of being general, we shall first turn again to the specific interconnection in Figure 17 and anticipate, on some examples, the general paradigm and tools that are available in robust control.

4.1 Uncertainties in the Tracking Configuration - Examples

4.1.1 A Classical SISO Example

This example serves as preparation towards the general approach to deal with additive uncertainties and as a review of Section 2.4.

Let us be concrete and assume that the model $G(s)$ in Figure 17 is given as

$$G(s) = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}. \tag{4.1}$$

Suppose the controller is chosen as

$$K(s) = \frac{0.1s+1}{(0.65s+1)(0.03s+1)}. \tag{4.2}$$

The code

```
s = zpk('s');
G = 200/(10*s + 1)/(0.05*s + 1)^2;
K = (0.1*s + 1)/(0.65*s + 1)/(0.03*s + 1);
systemnames='G';
inputvar='[d;n;r;u]';
outputvar='[G+d-r;r-n-d-G]';
input_to_G='[u]';
P = sysic;
S=lft(P,K);
[A,B,C,D]=ssdata(S);
eig(A)
```

actually computes realizations of the open-loop interconnection P, of the controller K, and of the closed-loop interconnection $P \star K$ denoted as (A, B, C, D). It turns out that A is stable such that K stabilizes P.

Suppose that we know (for example from frequency domain experiments) that the frequency response $H(i\omega)$ of the actual stable plant $H(s)$ does not coincide with that of the model $G(i\omega)$. Let us assume that we can even quantify this mismatch as

$$|H(i\omega) - G(i\omega)| < 1 \;\text{ for all }\; \omega \in \mathbb{R} \cup \{\infty\}. \tag{4.3}$$

Here is the fundamental question we would like to ask: If we replace G by H, does the controller still stabilize the feedback interconnection?

If we knew H, we could just plug in H and test this property in the same way as we did for G. Unfortunately, however, H could be any element of the *set* of all stable systems H that satisfy (4.3). Hence, in principle, we would have to test infinitely many transfer functions H what is not possible.

This motivates to look for alternative verifiable tests. Let us introduce the notation

$$\Delta(s) := H(s) - G(s)$$

for the plant-model mismatch. Then the actual plant is given as

$$H(s) = G(s) + \Delta(s)$$

with some stable $\Delta(s)$ that satisfies

$$|\Delta(i\omega)| < 1 \;\text{ for all }\; \omega \in \mathbb{R} \cup \{\infty\}. \tag{4.4}$$

Therefore, our main question can be formulated as follows: Does the closed-loop interconnections as depicted in Figure 24 remain stable if Δ is any stable transfer function that satisfies (4.4)?

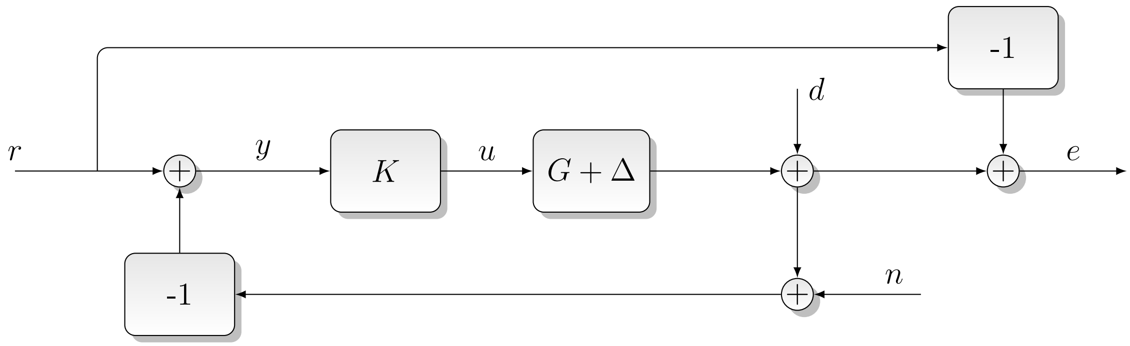


Figure 24: Uncertain closed-loop interconnection

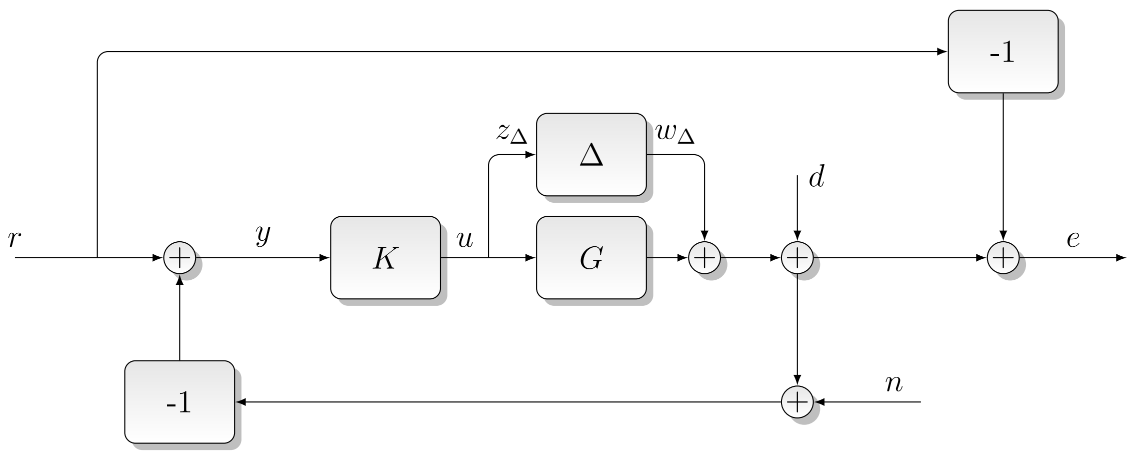


Figure 25: Rewritten uncertain closed-loop interconnection

We could also ask instead: Does there exists a stable $\Delta(s)$ with (4.4) that *destabilizes* the closed-loop interconnection?

Roughly speaking, the answer is obtained by looking at the influence which the uncertainty can exert on the interconnection. For that purpose we calculate the transfer function that is 'seen' by Δ: Just rewrite the loop as in Figure 25 in which we have just introduced notations for the input signal z_Δ and the output signal w_Δ of Δ. After this step we disconnect Δ to arrive at the interconnection in Figure 26. The transfer function seen by Δ is nothing but the transfer function $w_\Delta \to z_\Delta$.

For this specific interconnection, a straightforward calculation reveals that this transfer function is given as

$$M = -(I + KG)^{-1}K.$$

As a fundamental result, we will reveal that the loop remains stable for a specific Δ if $I - M\Delta$ does have a proper and stable inverse.

Let us motivate this result by putting the interconnection in Figure 25 into the the general

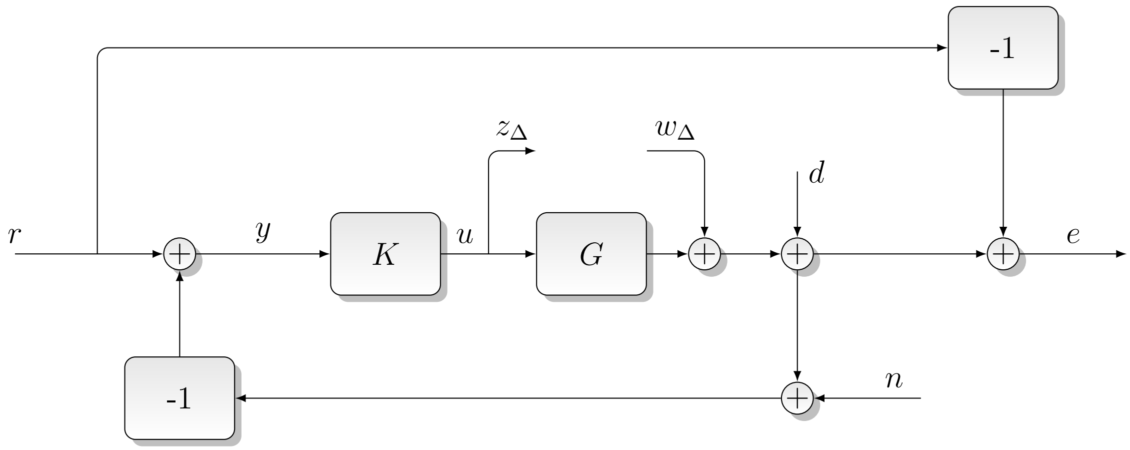


Figure 26: Closed-loop interconnection with disconnected uncertainty

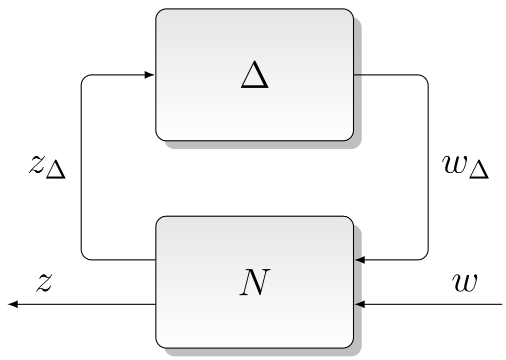


Figure 27: Uncertain closed-loop interconnection

structure as in Figure 27 by setting $z = e$ and collecting again all the signals d, n, r into the vector-valued signal $w = \begin{pmatrix} d \\ n \\ r \end{pmatrix}$ as we did previously. Then Figure 26 corresponds to Figure 28.

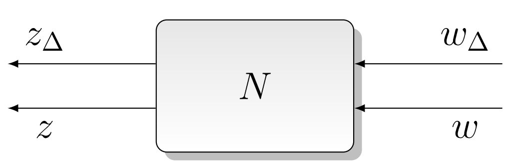


Figure 28: Uncertain closed-loop interconnection

Mathematically, the system in Figure 28 with disconnected uncertainty is described as

$$\begin{pmatrix} z_\Delta \\ z \end{pmatrix} = N \begin{pmatrix} w_\Delta \\ w \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w \end{pmatrix}$$

where N is partitioned according to the (possibly vector valued) signals w_Δ, w and z_Δ, z. Then the transfer matrix seen by Δ is nothing but $N_{11} = M$.

If we reconnect the uncertainty as

$$w_\Delta = \Delta z_\Delta,$$

we arrive at

$$z = [N_{22} + N_{21}\Delta(I - M\Delta)^{-1}N_{12}]w.$$

This easily clarifies the above statement: Since the controller is stabilizing, all $N_{11} = M$, N_{12}, N_{21}, N_{22} are proper and stable. Only through the inverse $(I - M\Delta)^{-1}$, improperness or instability might occur in the loop. Therefore, if $I - M\Delta$ does have a proper and stable inverse, the loop remains stable.

Note that these arguments are not sound: We did not proved stability of the interconnection as defined in Definition 3.2. We will provide rigorous arguments in Section 4.7.

What have we achieved for our specific interconnection? We have seen that we need to verify whether

$$I - M\Delta = I + (I + KG)^{-1}K\Delta$$

does have a proper stable inverse for all stable Δ with (4.4). Let us apply the Nyquist criterion: Since both $M = -(I + KG)^{-1}K$ and Δ are stable, this is true if the Nyquist curve

$$\omega \to -M(i\omega)\Delta(i\omega) = (I + K(i\omega)G(i\omega))^{-1}K(i\omega)\Delta(i\omega)$$

does not encircle the point -1. This is certainly true if

$$|M(i\omega)\Delta(i\omega)| = |(I + K(i\omega)G(i\omega))^{-1}K(i\omega)\Delta(i\omega)| < 1 \;\text{ for all }\; \omega \in \mathbb{R} \cup \{\infty\}. \tag{4.5}$$

Due to (4.4), this is in turn implied by the condition

$$|M(i\omega)| = |(I + K(i\omega)G(i\omega))^{-1}K(i\omega)| \leq 1 \;\text{ for all }\; \omega \in \mathbb{R} \cup \{\infty\}. \tag{4.6}$$

We conclude: If (4.6) is valid, the transfer function $I - M\Delta = I + (I + KG)^{-1}K\Delta$ does have a proper and stable inverse for all stable Δ with (4.4), and hence none of these uncertainties can destabilize the loop.

To continue with the example, the code

```
s = zpk('s');
G = 200/(10*s + 1)/(0.05*s + 1)^2;
K = (0.1*s + 1)/(0.65*s + 1)/(0.03*s + 1);
systemnames = 'G';
inputvar = '[w;d;n;r;u]';
outputvar = '[u;w+G+d-r;r-w-n-d-G]';
input_to_G = '[u]';
P = sysic;
N = lft(P, K);
[A, B, C, D] = ssdata(N);
eig(A)
M = N(1, 1);
om = logspace(-2, 2);
Mom = squeeze(freqresp(M, 1i*om));
loglog(om, abs(Mom));
grid on
```

determines the transfer matrix N, it picks out the left upper block M, the transfer function seen by the uncertainty, and plots the magnitude of M over frequency; the result is shown in Figure 29. Since the magnitude exceeds one at some frequencies, we see that we *cannot guarantee robust stability* against all stable Δ that satisfy (4.4).

Although this is a negative answer, the plot provides us with a lot of additional insight.

Let us first *construct* an uncertainty that destabilizes the loop. This is expected to happen for a Δ for which $(I-M\Delta)^{-1}$ has an unstable pole, i.e., for which $I-M\Delta$ has an unstable zero. Let us look specifically for a zero $i\omega_0$ on the imaginary axis; then we need to have

$$M(i\omega_0)\Delta(i\omega_0) = 1.$$

Let us pick ω_0 such that $|M(i\omega_0)| > 1$. As the magnitude plot shows, such a frequency indeed exists. Then the *complex number*

$$\Delta_0 := \frac{1}{M(i\omega_0)}$$

indeed renders $M(i\omega_0)\Delta_0 = 1$ satisfied. In our example, we chose $\omega_0 = 5$. If we calculate Δ_0 and replace G by $G+\Delta_0$, a state-space realization of the close-loop interconnection as calculated earlier will have an eigenvalue $5i$ and is, hence, unstable. We have constructed a complex number Δ_0 that destabilizes the interconnection. Note, however, that complex number are not in our uncertainty class that consisted of real rational proper transfer functions only. Lemma 2.14 helps to find such destabilizing perturbation from Δ_0.

In fact, Lemma 2.14 says that we can construct a real-rational proper and stable $\Delta(s)$ satisfying

$$\Delta(i\omega_0) = \Delta_0, \quad |\Delta(i\omega)| = |\Delta_0| < 1 \ \text{ for all } \ \omega \in \mathbb{R}\cup\{\infty\}.$$

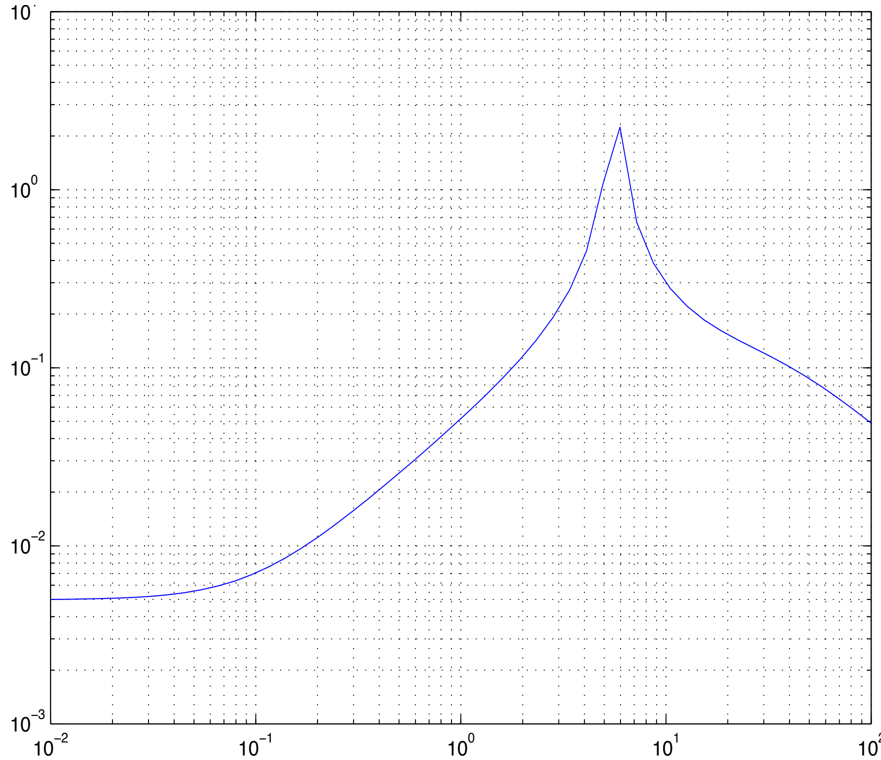


Figure 29: Magnitude plot of M

In our case the construction leads to $\alpha = -0.8434$ and $\beta = -4.6257$. As expected, the A matrix of a realization of the closed-loop interconnection for $G + \Delta$ turns out to have $5i$ as an eigenvalue. We have hence found a stable destabilizing uncertainty whose frequency response is smaller than 1.

To summarize, we have seen that the loop is not robustly stable against all the uncertainties in the class we started out with. What can we conclude on the positive side? In fact, Figure 29 shows that

$$|M(i\omega)| = |(I + K(i\omega)G(i\omega))^{-1}K(i\omega)| \leq 4 \text{ for all } \omega \in \mathbb{R}\{\infty\}.$$

Therefore, (4.5) holds for all stable Δ that satisfy

$$|\Delta(i\omega)| < \frac{1}{4} \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

Hence, we *can guarantee robust stability* for all uncertainty in this *smaller class*.

In fact, the largest bound r for which we can still guarantee robust stability for any stable Δ satisfying

$$|\Delta(i\omega)| < r \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}$$

is given by the reciprocal of the peak value of the magnitude plot:

$$r = \left(\sup_{\omega \in \mathbb{R} \cup \{\infty\}} |M(i\omega)| \right)^{-1} = \|M\|_{\infty}^{-1}.$$

We have discussed with this simple example how one can test robust stability by looking at a magnitude plot of the transfer function 'seen' by Δ. If robust stability does not hold, we have discussed how to construct a destabilizing perturbation.

4.1.2 A Modern MIMO Example

In the last section we have considered a very elementary example of a feedback interconnection in which only one uncertainty occurs.

Let us hence look at a model that is described by the 2×2 transfer matrix

$$G(s) = \frac{1}{s^2 + a^2} \begin{pmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{pmatrix}$$

with minimal state-space realization

$$G = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right].$$

Suppose that this is a model of a system in which certain tolerances for the actuators have to be taken into account that are represented by parametric uncertainties. Let us hence assume that the input matrix is rather given by

$$\begin{pmatrix} 1 + \delta_1 & 0 \\ 0 & 1 + \delta_2 \end{pmatrix}.$$

Hence the actual system is

$$\frac{1}{s^2 + a^2} \begin{pmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{pmatrix} \left(I + \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \right)$$

or

$$G(I + \Delta) \text{ with } \Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$$

for some real numbers δ_1, δ_2 with

$$|\delta_1| < r, \ |\delta_2| < r. \tag{4.7}$$

Again, we are faced with a whole set of systems rather than with a single one. Uncertainty now enters via the two real parameters δ_1, δ_2.

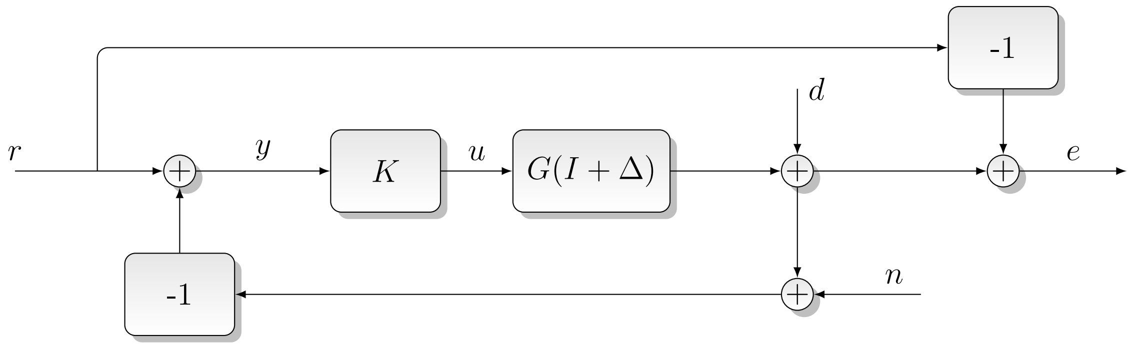


Figure 30: Uncertain closed-loop interconnection

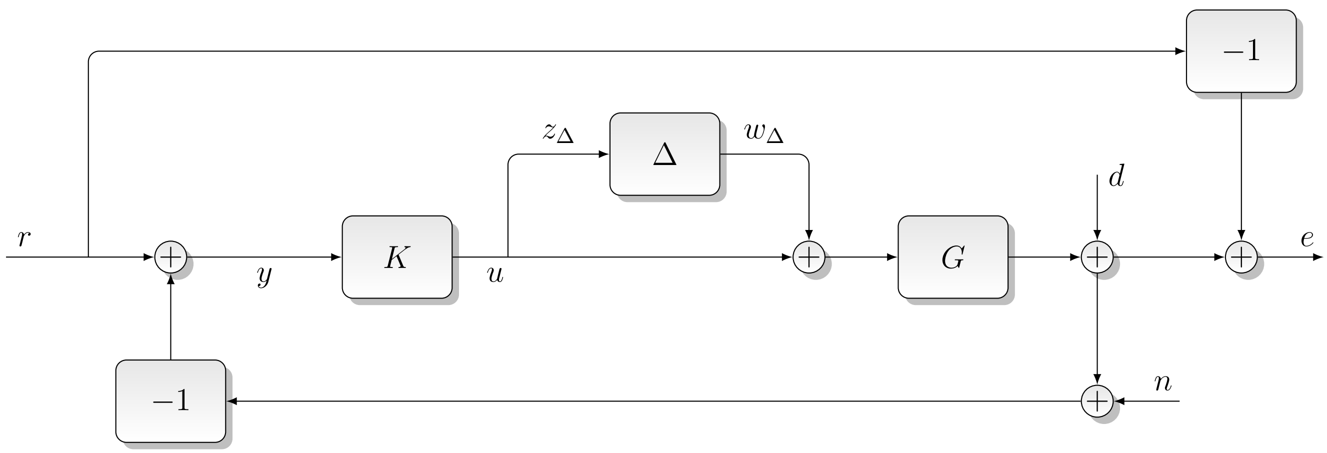


Figure 31: Rewritten uncertain closed-loop interconnection

Let us take the unity feedback controller

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and consider again the interconnection in Figure 30.

As before we rewrite the interconnection in the obviously equivalent fashion as in Figure 31 and disconnect the uncertainty as in Figure 32.

The transfer matrix seen by Δ is given as

$$M(s) = \frac{1}{s+1} \begin{pmatrix} -1 & -a \\ a & -1 \end{pmatrix}. \tag{4.8}$$

As indicated earlier and as it will be developed in the general theory, for testing robust stability we have to verify whether $I - M\Delta$ has a proper and stable inverse for all Δ.

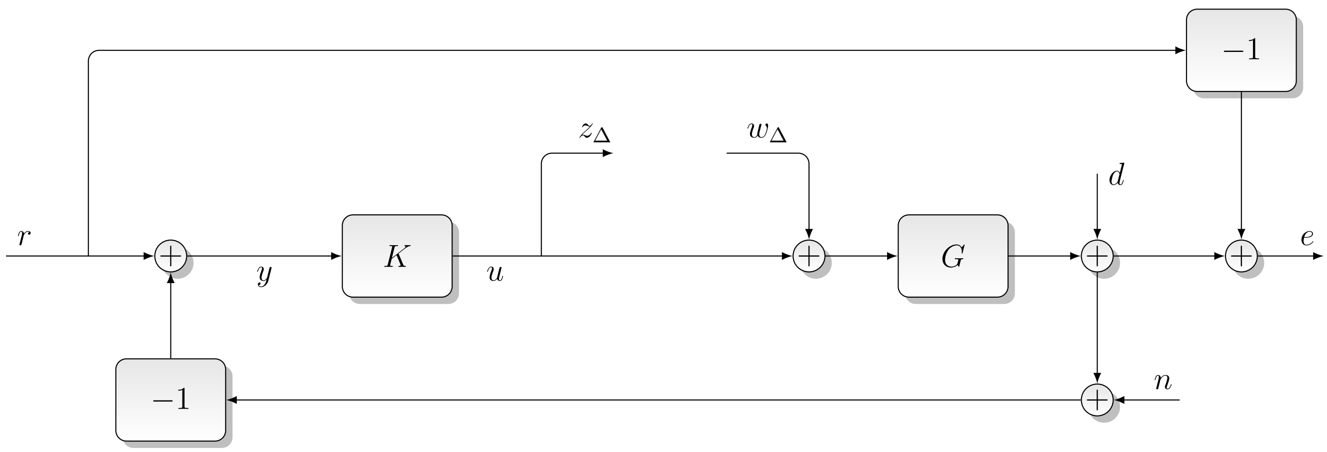


Figure 32: Rewritten uncertain closed-loop interconnection

Recall that, by Lemma 1.4, this is true iff $I - M(s)\Delta(s)$ is non-singular for $s = \infty$ (properness, no pole at infinity) and for all s in the closed right-half plane (stability, no pole in closed right-half plane). Hence we have to check whether the determinant is non-zero for all $s \in \mathbb{C}_= \cup \mathbb{C}_> \cup \{\infty\}$. The determinant of

$$I - M(s)\Delta = \begin{pmatrix} 1 + \frac{\delta_1}{s+1} & \frac{a\delta_2}{s+1} \\ \frac{-a\delta_1}{s+1} & 1 + \frac{\delta_2}{s+1} \end{pmatrix}$$

is easily calculated to

$$\frac{1}{(s+1)^2}\left(s^2 + (2 + \delta_1 + \delta_2)s + (1 + \delta_1 + \delta_2) + (a^2 + 1)\delta_1\delta_2\right).$$

It does not have a zero at ∞. Moreover, its finite zeros are certainly confined to $\mathbb{C}_<$ if and only if

$$2 + \delta_1 + \delta_2 > 0, \quad (1 + \delta_1 + \delta_2) + (a^2 + 1)\delta_1\delta_2 > 0.$$

For $a = 10$, Figure 33 depicts the region of parameters where this condition is *not true*.

Let us now first concentrate on one uncertainty at a time. For $\delta_2 = 0$, the stability conditions holds for all δ_1 in the big interval $(-1, \infty)$. The same holds for $\delta_1 = 0$ and $\delta_2 \in (-1, \infty)$.

Let us now vary both parameters. If we try to find the largest r such that stability is preserved for all parameters with (4.7), we have to inflate a box around zero until it hits the region of instability as shown in Figure 33. (Why?) For the parameter $a = 10$, the largest r turns out to be 0.1, and this value *shrinks* with increasing a.

In summary, the analysis with a single varying uncertainty ($\delta_1 = 0$ or $\delta_2 = 0$) just gives a wrong picture of the robust stability region for *common* variations.

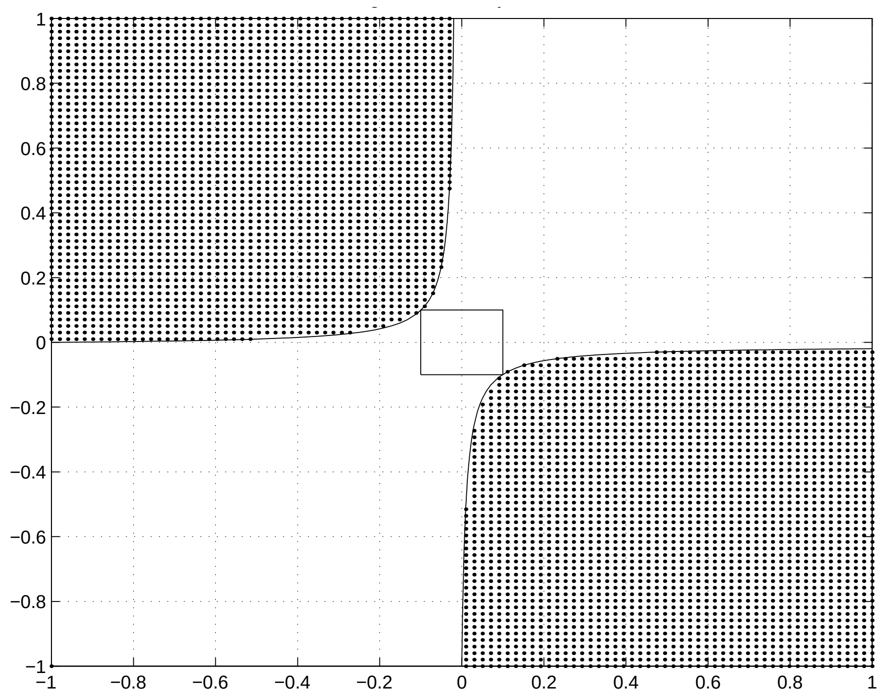


Figure 33: Dotted: Region of destabilizing parameters (δ_1, δ_2)

It is important to observe that we could very easily explicitly determine the region of stability and instability for this specific problem. Since this is by no means possible in general, we need to have a general tool that can be applied as well to arrive at similar insights for more sophisticated structures. This is the goal in the theory to be developed in the next sections.

4.2 Types of Uncertainties of System Components

Uncertainties that can be dealt with by the theory to be developed include parametric and LTI dynamic uncertainties. Parametric uncertainties are related to variations of real parameters (mass, spring constants, damping,...) in a system model, whereas LTI dynamic uncertainty should capture unmodeled dynamics of a system.

4.2.1 Parametric Uncertainties

Consider the simple mechanical system depicted in Figure 34 modeled as

$$m_1\ddot{\xi}_1 = u - f, \quad m_2\ddot{\xi}_2 = f, \quad f = k(\xi_1 - \xi_2), \quad y = \xi_2$$

with nominal values $m_1 = m_2 = k = 1$.

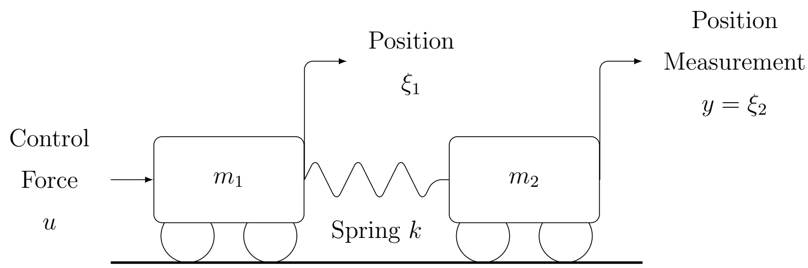


Figure 34: Simple mechanical system.

In the classical feedback interconnection (Figure 17) the controller

$$K(s) = \frac{29.434(s + 0.2329)(s^2 - 0.7687s + 2.917)}{(s + 9.843)(s + 2.227)(s^2 + 1.81s + 4.063)}$$

is stabilizing if m_1, m_2, k are precisely known, which might be unrealistic.

Let us now suppose that m_1, m_2, k are only known within 20% of their nominal value, which means that

$$m_1 \in [0.8, 1.2], \quad m_2 \in [0.8, 1.2], \quad k \in [0.8.1.2].$$

Each parameter triple defines another system. Hence, effectively, we consider a whole family of systems. A fundamental question:

Is the controller **robustly stabilizing**: Does it internally stabilize all possible systems in our family?

The Robust Control Toolbox offers a very simple approach in order to handle parametric uncertainties. Instead of defining parameters as real numbers, just define them as **uncertain atom objects** (command `ureal`) and build your interconnection in the same fashion as before. The result is an **uncertain system object**. With `usubs` one can then easily pick one particular system (or with `usample` a randomly chosen sample) out of the whole family, and then test whether K stabilizes the resulting finitely many systems:

```
m1 = ureal('m1', 1, 'percent', 20);
m2 = ureal('m2', 1, 'percent', 20);
k = ureal('k', 1, 'percent', 20);
s = zpk('s');
G1 = 1/(m1 * s^2);
G2 = 1/(m2 * s^2);
systemnames = 'G1 G2 k';
inputvar = '[u]';
outputvar = '[G2]';
```

```
input_to_G1 = '[u - k]';
input_to_G2 = '[k]';
input_to_k = '[G1 - G2]';
G = sysic; % builds uncertain subsystem

systemnames = 'G';
inputvar = '[d; n; r; u]';
outputvar = '[G + d - r; r - n - d - G]';
input_to_G = '[u]';
olic = sysic; % builds uncertain interconnection

% nominal stabilizing controller
K = 29.434 * (s + 0.2329) * (s^2 - 0.7687*s + 2.917) / (s + 9.843) / ...
    (s + 2.227) / (s^2 + 1.81*s + 4.063);

clic = lft(olic, K);
cl0 = usubs(clic, 'm1', 1, 'm2', 1, 'k', 1); % nominal system
clr = usample(clic, 10); % some samples
max(real(eig(cl0)));
plot(squeeze(eig(clr)),'*');
```

Many familiar commands for LTI systems, like `bode`, are overlayed for uncertain systems. Typically, the operation is performed for the nominal system and a specific number of random samples ($1 + 20$ for `bode`). This provides a good intuition of sensitivity against parameter variations.

In our example the closed loop poles for internal stability resulting from the above code are depicted in Figure 36. However the samples of bode diagramms and step responses of the transfer matrix from reference to error depicted in Figure 35 indicate that the controller K does not internally stabilize all possible systems in our family.

4.2.2 Dynamic Uncertainties

One can estimate the frequency response of a real stable SISO plant by injecting sinusoidal signals. If performing measurements at one frequency, one does usually not obtain just one complex number that could be taken as an estimate for the plant's response at frequency ω, but, instead, it's a whole set of complex numbers that is denoted by $\mathcal{H}(\omega)$. Such an experiment would lead us to the conclusion that any proper and stable $H(s)$ that satisfies

$$H(i\omega) \in \mathcal{H}(\omega)$$

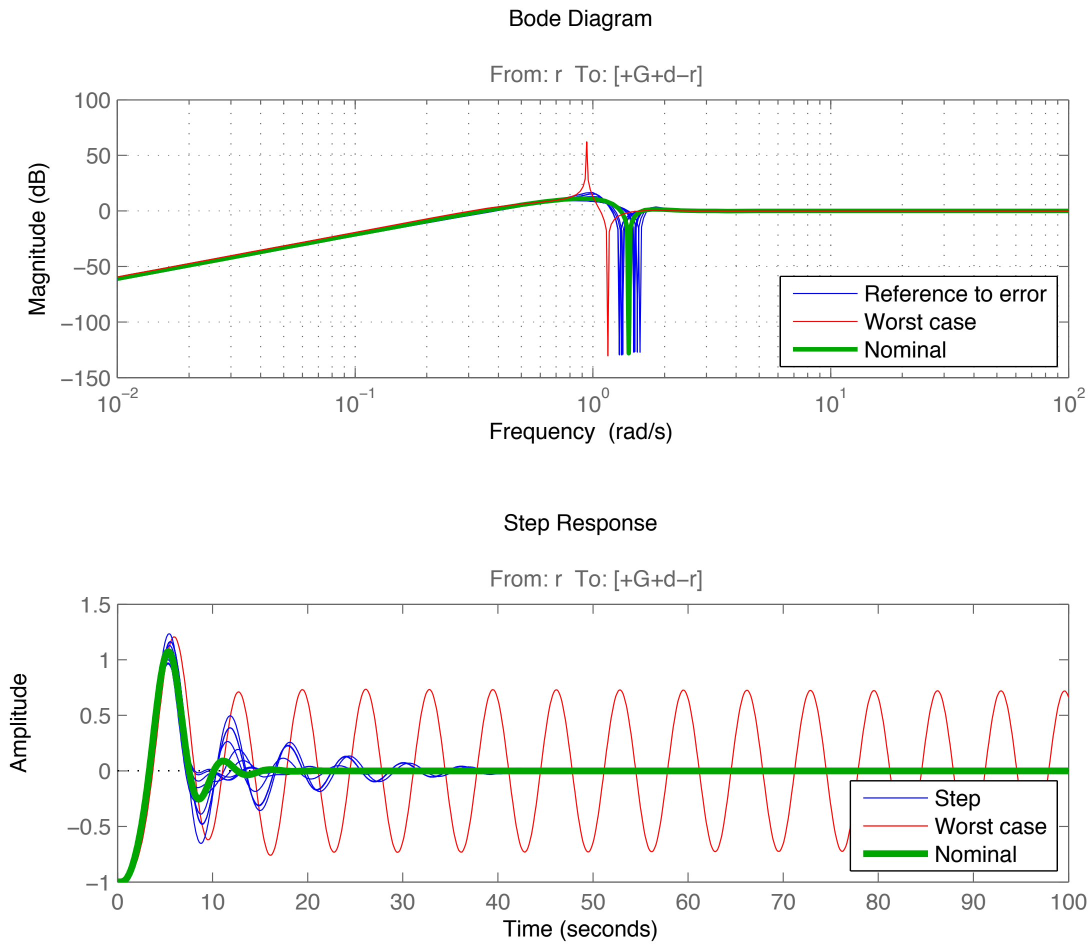


Figure 35: Samples of bode diagramms and step responses of a system affected by parametric uncertainties

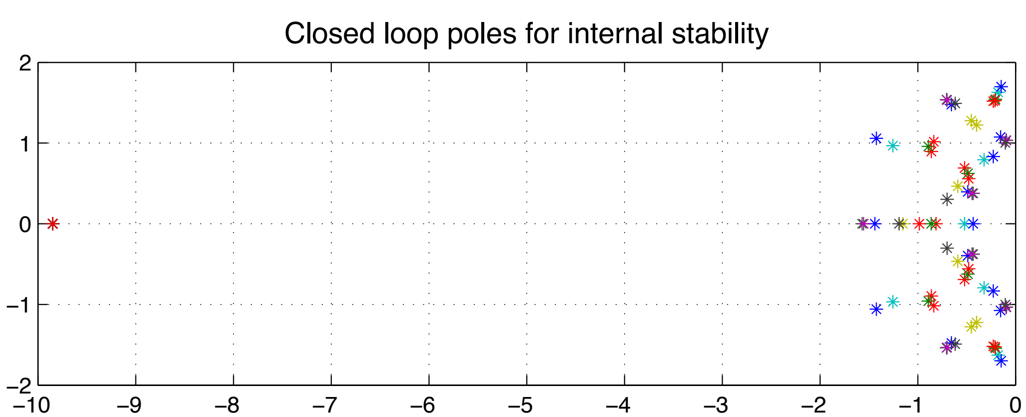


Figure 36: Closed loop poles for internal stability of a system affected by parametric uncertainties

is an appropriate model for the underlying plant. Since one can only perform a finite number of measurements, $\mathcal{H}(\omega)$ is usually only available at finitely many frequencies and consists of finitely many points. Due to the lack of a nice description, this set is not appropriate for the theory to be developed.

Hence we try to *cover* $\mathcal{H}(\omega)$ with a set that admits a more appropriate description. This means

$$\mathcal{H}(\omega) \subset G(i\omega) + W(i\omega)\mathbf{\Delta}_c \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\},$$

where

- $G(s)$ is a real rational proper transfer matrix
- $\mathbf{\Delta}_c$ is the open unit disk around 0: $\mathbf{\Delta}_c := \{\Delta_c \in \mathbb{C} \mid |\Delta_c| < 1\}$
- $W(s)$ is a real rational weighting function.

At each frequency we have hence covered the unstructured set $\mathcal{H}(\omega)$ with the disk

$$G(i\omega) + W(i\omega)\mathbf{\Delta}_c \tag{4.9}$$

whose center is $G(i\omega)$ and whose radius is $|W(i\omega)|$. In this description, G admits the interpretation as a nominal system. The deviation from $G(i\omega)$ is given by the circle $W(i\omega)\mathbf{\Delta}_c$ whose radius $|W(i\omega)|$ varies with frequency. Hence, the weighting function W captures how the size of the uncertainties depends upon the frequency; this allows to take into account that models are, usually, not very accurate at high frequency; typically, W is a high-pass filter.

Note that we proceeded similarly as in the parametric case: At frequency ω, we represent the deviation by a nominal value $G(i\omega)$ and by a $W(i\omega)$-weighted version of the open unit disk.

The actual set of uncertainties is then defined as

$$\mathbf{\Delta} := \{\Delta(s) \in RH_\infty \mid \Delta(i\omega) \in \mathbf{\Delta}_c \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}\}, \tag{4.10}$$

the set of all proper and stable transfer functions that take their values along the imaginary in the open unit disk. Note that this set is nothing but

$$\{\Delta(s) \in RH_\infty \mid \|\Delta\|_\infty < 1\} \tag{4.11}$$

which is often called the open unit ball in RH_∞. (It is important to digest that (4.10) and (4.11) are just the same!)

Finally, the uncertain system is described by

$$G_\Delta := G + W\Delta \ \text{ with } \ \Delta \in \mathbf{\Delta}.$$

As for real uncertainties, we have obtained a whole set of systems that is now parameterized by the uncertain dynamics Δ in $\mathbf{\Delta}$.

Remark 4.1

1) The set of values $\mathbf{\Delta}_c$ must not necessarily be a circle for our results to apply. It can be an arbitrary set that contains 0, such as a polytope. The required technical hypothesis are discussed in Section 4.5. The deviation set

$$W(i\omega)\mathbf{\Delta}_c \tag{4.12}$$

is then obtained by shrinking/stretching $\mathbf{\Delta}_c$ with factor $|W(i\omega)|$, and by rotating it according to the phase of $W(i\omega)$.

2) We could be even more general and simply allow for frequency dependent value sets $\mathbf{\Delta}_c(\omega)$ that are not necessarily described as (4.12). Then we can more accurately incorporate phase information about the uncertainty.

4.2.3 Mixed Uncertainties

Of course, in a certain system component, one might encounter both parametric and dynamic uncertainties. As an example, suppose that the diagonal elements of

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+3} & \frac{1}{2s+1} \end{pmatrix} \tag{4.13}$$

are not affected by uncertainties, but the numerator 1 of the right upper element is affected by perturbations such it actually equals

$$\frac{1+W_1\Delta_1}{s+2} = \frac{1}{s+2} + \frac{W_1}{s+2}\Delta_1 \quad \text{where} \quad |\Delta_1| < 1$$

and the left lower element equals

$$\frac{1}{s+3}(1+W_2(s)\Delta_2(s)) \quad \text{where} \quad \|\Delta_2\|_\infty < 1.$$

Here W_1 is a constant weight, $W_2(s)$ is a real rational weighting function, and Δ_1 is a parametric uncertainty in the unit interval $(-1, 1)$, whereas $\Delta_2(s)$ is a (proper stable) dynamic uncertainty that takes its values $\Delta(i\omega)$ on the imaginary axis in the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

Hence, the uncertain system is described as

$$G_\Delta(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+3} & \frac{1}{2s+1} \end{pmatrix} + \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2(s) \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{s+2}W_1 \\ \frac{1}{s+3}W_2(s) & 0 \end{pmatrix} \tag{4.14}$$

where $\Delta_1 \in \mathbb{R}$ is bounded as $|\Delta_1| < 1$, and $\Delta_2 \in RH_\infty$ is bounded as $\|\Delta_2\|_\infty < 1$.

This amounts to

$$G_\Delta(s) = G(s) + \Delta(s)W(s)$$

with a nominal system $G(s)$, a matrix valued weighting $W(s)$, and *block-diagonally strcutured*

$$\Delta(s) = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2(s) \end{pmatrix}.$$

Note also that the diagonal blocks of $\Delta(s)$ have a different nature (Δ_1 is real, Δ_2 is dynamic) and they are bounded in size over frequency, where the bound for both is rescaled to 1 by using weighting functions.

All these properties of $\Delta(s)$ (structure and bound on size) can be captured by simply specifying a set of values that consists of complex matrices as follows:

$$\mathbf{\Delta}_c := \{ \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \mid \Delta_1 \in \mathbb{R},\ |\Delta_1| < 1,\ \Delta_2 \in \mathbb{C},\ |\Delta_2| < 1\}.$$

The set of uncertainties $\Delta(s)$ is, again, just given by (4.10). We have demonstrated the flexibility of the abstract setup (4.10) if we allow for subsets $\mathbf{\Delta}_c$ of matrices.

To conclude, we have brought the specific example back to the same general scheme: We have parameterized the actual set of systems G_Δ as $G + \Delta W$ where, necessarily, W has to be chosen as a matrix, and the uncertainty $\Delta \in \mathbf{\Delta}$ turns out to admit a *block-diagonal structure*.

Remark 4.2 As mentioned previously, we could considerably increase the flexibility in uncertainty modeling if not only allowing to constrain the elements of the matrices in $\mathbf{\Delta}_c$ by disks or real intervals; under the technical hypotheses as discussed in Section 4.5, all the results to follow still remain valid.

4.2.4 Unstructured Uncertainties

Let us again consider the plant model $G(s)$ in (4.13). Suppose this is an accurate model at low frequencies, but it is known that the accuracy of *all entries* decreases at high frequencies. With a (real rational) SISO high-pass filter $W(s)$, the actual frequency response is rather described as

$$G(i\omega) + W(i\omega)\Delta_c$$

where

$$\Delta_c = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}$$

is any complex matrix that is bounded as

$$\|\Delta_c\| = \sigma_{\max}(\Delta_c) < 1.$$

Obviously, the size of the deviation $W(i\omega)\Delta_c$ at frequency ω is bounded as

$$\|W(i\omega)\Delta_c\| = |W(i\omega)|\|\Delta_c\| \le |W(i\omega)|.$$

(If $|W(i\omega)|$ is not zero, we have a strict inequality.) Hence, the frequency dependence of the size is captured by the dependence of $|W(i\omega)|$ on ω. Note that this behavior is the same for all the elements of the 2×2 matrix deviation $W(i\omega)\Delta_c$.

We chose the maximal singular value to evaluate the size of the matrix $W(i\omega)\Delta_c$ since this is an appropriate measure for the gain of the error at this frequency, and since the theory to be developed will then turn out more satisfactory than for other choices.

Let us now subsume this specific situation in our general scenario: We take

$$\boldsymbol{\Delta}_c := \{\Delta_c \in \mathbb{C}^{2\times 2} \mid \|\Delta_c\| < 1\}$$

and describe the uncertain system as

$$G + W\Delta$$

where $\Delta \in \boldsymbol{\Delta}$ as given in (4.10). Since $\boldsymbol{\Delta}_c$ is a set of full matrices without any specific structural aspects, this type of uncertainty is called *unstructured* or a *full-block uncertainty.*

4.2.5 Unstructured versus Structured Uncertainties

Continuing with the latter example, we might know more about the individual deviation of each element of $G(s)$. Suppose that, at frequency ω, the actual model is

$$G(i\omega) + \begin{pmatrix} W_{11}(i\omega)\Delta_{11} & W_{12}(i\omega)\Delta_{12} \\ W_{21}(i\omega)\Delta_{21} & W_{22}(i\omega)\Delta_{22} \end{pmatrix} \tag{4.15}$$

where the complex numbers Δ_{jk} satisfy

$$|\Delta_{11}| < 1, \;\; |\Delta_{12}| < 1, \;\; |\Delta_{21}| < 1, \;\; |\Delta_{22}| < 1, \tag{4.16}$$

and the real rational (usually high-pass) SISO transfer functions $W_{11}(s)$, $W_{12}(s)$, $W_{21}(s)$, $W_{22}(s)$ capture the variation of size over frequency as in our SISO examples.

We could rewrite (4.15) as

$$G(i\omega) + \begin{pmatrix} W_{11}(i\omega) & 0 & W_{12}(i\omega) & 0 \\ 0 & W_{21}(i\omega) & 0 & W_{22}(i\omega) \end{pmatrix} \begin{pmatrix} \Delta_{11} & 0 \\ \Delta_{21} & 0 \\ 0 & \Delta_{12} \\ 0 & \Delta_{22} \end{pmatrix}$$

or as

$$G(i\omega) + \begin{pmatrix} W_{11}(i\omega) & 0 & W_{12}(i\omega) & 0 \\ 0 & W_{21}(i\omega) & 0 & W_{22}(i\omega) \end{pmatrix} \begin{pmatrix} \Delta_{11} & 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 & 0 \\ 0 & 0 & \Delta_{12} & 0 \\ 0 & 0 & 0 & \Delta_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Obviously, we have to live with *structure* in the uncertainty if we would like to use different weightings for the various elements in the uncertainty. However, the two displayed structures differ: In the first case, we have two 2×1 blocks on the diagonal, whereas in the second one we have four 1×1 blocks on the diagonal. What should we choose for?

As mentioned above, we would like to take as a measure of size of Δ_c the largest singular value; in fact, Δ_c is supposed to be bounded as $\|\Delta_c\| < 1$. Then we observe that

$$\left\| \begin{pmatrix} \Delta_{11} & 0 \\ \Delta_{21} & 0 \\ 0 & \Delta_{12} \\ 0 & \Delta_{22} \end{pmatrix} \right\| < 1$$

is equivalent to

$$\left\| \begin{pmatrix} \Delta_{11} \\ \Delta_{21} \end{pmatrix} \right\| < 1, \ \left\| \begin{pmatrix} \Delta_{12} \\ \Delta_{22} \end{pmatrix} \right\| < 1 \ \text{ or } \ |\Delta_{11}|^2 + |\Delta_{21}|^2 < 1 \ \text{ and } \ |\Delta_{12}|^2 + |\Delta_{22}|^2 < 1.$$

This is *not what we want.* If we insist on (4.16), we have to work with the second structure since

$$\left\| \begin{pmatrix} \Delta_{11} & 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 & 0 \\ 0 & 0 & \Delta_{12} & 0 \\ 0 & 0 & 0 & \Delta_{22} \end{pmatrix} \right\| < 1$$

is equivalent to (4.16).

4.3 Summary on Uncertainty Modeling for Components

For MIMO models of system components, we can work with only rough descriptions of modeling errors. Typically, the uncertain component is described as

$$G_\Delta = G + W_1 \Delta W_2 \tag{4.17}$$

with real rational weighting matrices W_1 and W_2 and full block or unstructured uncertainties Δ that belongs to $\mathbf{\Delta}$ as defined in (4.10) where

$$\mathbf{\Delta}_c := \{\Delta_c \in \mathbb{C}^{p\times q} \mid \|\Delta_c\| < 1\}.$$

If we choose for a more refined description of the uncertainties, the uncertainties in (4.17) will admit a certain *structure* that is often *block-diagonal*. To be specific, the uncertainty set $\mathbf{\Delta}$ will be given by (4.10) with

$$\mathbf{\Delta}_c := \left\{ \Delta_c = \begin{pmatrix} \delta_1 & & & & & 0 \\ & \ddots & & & & \\ & & \delta_r & & & \\ & & & \Delta_1 & & \\ & & & & \ddots & \\ 0 & & & & & \Delta_f \end{pmatrix} \mid \delta_j \in \mathbb{R}, \quad \Delta_j \in \mathbb{C}^{p_j \times q_j}, \quad \|\Delta_c\| < 1 \right\}.$$

Note that we have distinguished the *real blocks* δ_j that correspond to parametric uncertainties from the *complex blocks* that are related to dynamic uncertainties. Note also that

$$\|\Delta_c\| < 1 \;\; \text{just means} \;\; |\delta_j| < 1, \quad \|\Delta_j\| < 1.$$

The weighting matrices W_1 and W_2 capture the variation of the uncertainty with frequency, and they determine how each of the blocks of the uncertainty appears in G_Δ.

Finally, we have seen that there is a lot of flexibility in the choice of the structure. It is mainly dictated by how refined one wishes to describe the individual uncertainties that appear in the model.

4.4 Pulling out the Uncertainties

As we have seen in the example, a central ingredient in testing robust stability is to calculate the transfer matrix 'seen' by Δ in an open-loop interconnection. We would like to explain how one can systematically perform these calculations.

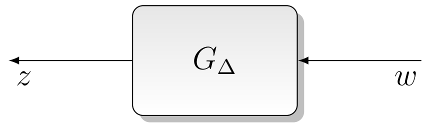


Figure 37: Uncertain Component

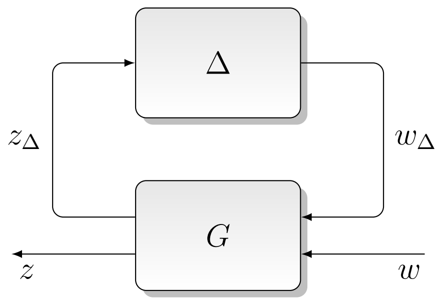


Figure 38: Uncertainty Pulled out of Component

4.4.1 Pulling Uncertainties out of Subsystems

First we observe that an interconnection is usually built from subsystems. These subsystems themselves might be subject to uncertainties. Hence we assume that they are parameterized as G_Δ with $\Delta \in \boldsymbol{\Delta}$ as shown in Figure 37.

In order to proceed one has to rewrite this system in the form as shown in Figure 38.

As an illustration, look at (4.14). This system can be rewritten as

$$z = (G_{22} + G_{21}\Delta G_{12})w$$

for

$$G_{22}(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+3} & \frac{1}{2s+1} \end{pmatrix}, \quad G_{21}(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_{12}(s) = \begin{pmatrix} 0 & \frac{1}{s+2}W_1 \\ \frac{1}{s+3}W_2(s) & 0 \end{pmatrix}.$$

If we define

$$G = \begin{pmatrix} 0 & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

we observe that (4.14) is rewritten as

$$\begin{pmatrix} z_\Delta \\ z \end{pmatrix} = G \begin{pmatrix} w_\Delta \\ w \end{pmatrix}, \quad w_\Delta = \Delta z_\Delta$$

which are the algebraic relations that correspond to Figure 38.

The step of rewriting a system as in Figure 37 into the structure of Figure 38 is most easily performed if just introducing extra signals that enter and leave the uncertainties. We illustrate this technique with examples that are of prominent importance:

- **Additive uncertainty**:

$$z = (G+\Delta)w \quad \Longleftrightarrow \quad \begin{pmatrix} z_1 \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & G \end{pmatrix} \begin{pmatrix} w_1 \\ w \end{pmatrix}, \quad w_1 = \Delta z_1.$$

- **Input multiplicative uncertainty**.

$$z = G(I+\Delta)w \quad \Longleftrightarrow \quad \begin{pmatrix} z_1 \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ G & G \end{pmatrix} \begin{pmatrix} w_1 \\ w \end{pmatrix}, \quad w_1 = \Delta z_1.$$

- **Input-output multiplicative uncertainty**.

$$z = (I+\Delta_1)G(I+\Delta_2)w$$

  is equivalent to

$$\begin{pmatrix} z_1 \\ z_2 \\ z \end{pmatrix} = \begin{pmatrix} 0 & G & G \\ 0 & 0 & I \\ I & G & G \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w \end{pmatrix}, \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

  Let us show explicitly how to proceed in this example. Observe that $z = (I + \Delta_1)G(I + \Delta_2)w$ can be written as

$$z = G(I+\Delta_2)w + w_1,\ z_1 = G(I+\Delta_2)w,\ w_1 = \Delta_1 z_1.$$

  We have pulled out Δ_1. In a second step, we do the same with Δ_2. The above relations are equivalent to

$$z = Gw + Gw_2 + w_1, z_2 = w,\ z_1 = Gw + Gw_2,\ w_1 = \Delta_1 z_1,\ w_2 = \Delta_2 z_2.$$

  The combination into matrix relations leads to the desired representation.

- **Factor uncertainty**. Let G_2 have a proper inverse. Then

$$z = (G_1+\Delta_1)(G_2+\Delta_2)^{-1}w$$

  is equivalent to

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = \begin{pmatrix} 0 & -G_2^{-1} & G_2^{-1} \\ I & -G_1G_2^{-1} & G_1G_2^{-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w \end{pmatrix}, \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} z_1.$$

Again we perform the calculations for the purpose of illustration. We observe that $z = (G_1 + \Delta_1)(G_2 + \Delta_2)^{-1}w$ is nothing but

$$z = (G_1 + \Delta_1)\xi,\ (G_2 + \Delta_2)\xi = w$$

what can be rewritten as

$$z = G_1\xi + w_1,\ z_1 = \xi,\ G_2\xi + w_2 = w,\ w_1 = \Delta_1 z_1,\ w_2 = \Delta_2 z_1.$$

If we eliminate ξ via $\xi = G_2^{-1}(w - w_2)$, we arrive at

$$z = G_1G_2^{-1}w - G_1G_2^{-1}w_2 + w_1,\ z_1 = G_2^{-1}w - G_2^{-1}w_2,\ w_1 = \Delta_1 z_1,\ w_2 = \Delta_2 z_1.$$

Note that the manipulations are representatives of how to pull out the uncertainties, in particular if they occur in the denominator as it happens in factor uncertainty. It is often hard to pull out the uncertainties if just performing matrix manipulations. If we rather use the input-output representations of systems *including the signals*, this technique is often pretty straightforward to apply. As a general rule, blocks that occur in a *rational* fashion can be pulled out. Finally, let us note that all these manipulations can also be performed directly for a state-space description where the state and its derivative are as well viewed as a signal.

Again, we include an example. Let

$$\dot{x} = \begin{pmatrix} -1+\delta_1 & 2 \\ -1 & -2+\delta_2 \end{pmatrix} x$$

denote a system with real uncertain parameters. This system can be written as

$$\dot{x} = Ax + Bw,\ z = Cx,\ w = \Delta z$$

with

$$A = \begin{pmatrix} -1 & 2 \\ -1 & -2 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}.$$

Somewhat more general, suppose that

$$\dot{x} = A(\delta)x + B(\delta)w,\ z = C(\delta)x + D(\delta)w$$

where the matrices $A(.)$, $B(.)$, $C(.)$, $D(.)$ depend affinely on the parameter $\delta = \begin{pmatrix} \delta_1 & \cdots & \delta_k \end{pmatrix}$. This just means that they can be represented as

$$\begin{pmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{pmatrix} = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} + \sum_{j=1}^{k} \delta_j \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}.$$

Let us factorize

$$\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} = \begin{pmatrix} L_j^1 \\ L_j^2 \end{pmatrix} \begin{pmatrix} R_j^1 & R_j^2 \end{pmatrix} \tag{4.18}$$

where $\begin{pmatrix} L_j^1 \\ L_j^2 \end{pmatrix}$ and $\begin{pmatrix} R_j^1 & R_j^2 \end{pmatrix}$ have full column and row rank respectively. The original system can be described as

$$\begin{pmatrix} \dot{x} \\ z \\ \hline z_1 \\ \vdots \\ z_k \end{pmatrix} = \left(\begin{array}{cc|ccc} A_0 & B_0 & L_1^1 & \cdots & L_k^1 \\ C_0 & D_0 & L_1^2 & \cdots & L_k^2 \\ \hline R_1^1 & R_2^2 & 0 & & 0 \\ \vdots & & & \ddots & \\ R_k^1 & R_k^2 & 0 & & 0 \end{array}\right) \begin{pmatrix} x \\ w \\ \hline w_1 \\ \vdots \\ w_k \end{pmatrix}, \quad \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} \delta_1 I & & 0 \\ & \ddots & \\ 0 & & \delta_k I \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$$

where the sizes of the identity block in $\delta_j I$ is equal to the number of columns or rows of $\begin{pmatrix} L_j^1 \\ L_j^2 \end{pmatrix}$ or $\begin{pmatrix} R_j^1 & R_j^2 \end{pmatrix}$ respectively.

Remark. We have chosen the factorization (4.18) such that the number of columns/rows of the factors are minimal. This renders the size of the identity blocks in the uncertainty minimal as well. One could clearly work with an arbitrary factorization; then, however, the identity blocks will be larger and the representation is not as efficient as possible.

Again, we remark that we can represent any

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = \begin{pmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

with elements that are *rational* functions (quotients of polynomials) of $\delta = \begin{pmatrix} \delta_1 & \cdots & \delta_k \end{pmatrix}$ without pole at $\delta = 0$ by

$$\begin{pmatrix} \dot{x} \\ \hline z \\ z_\Delta \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \begin{pmatrix} x \\ \hline w \\ w_\Delta \end{pmatrix}, \quad w_\Delta = \Delta z_\Delta$$

where

$$\Delta = \begin{pmatrix} \delta_1 I & & 0 \\ & \ddots & \\ 0 & & \delta_k I \end{pmatrix}.$$

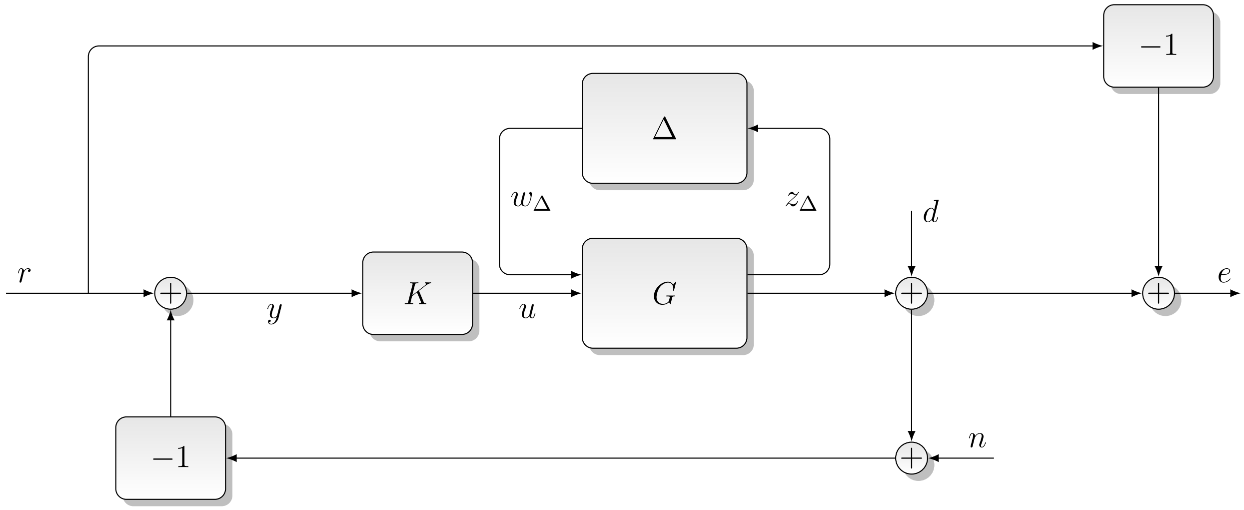


Figure 39: Uncertain closed-loop interconnection

We observe that Δ has a block-diagonal structure. Each block is given as

$$\delta_j I = \begin{pmatrix} \delta_j & & 0 \\ & \ddots & \\ 0 & & \delta_j \end{pmatrix}$$

and is said to be a real ($\delta_j \in \mathbb{R}$) block that is repeated if the dimension of the identity matrix is at least two.

4.4.2 Pulling Uncertainties out of Interconnections

We have seen various possibilities how to represent (37) as (38) for components. Let us now suppose this subsystem is part of a (possibly large) interconnection.

Again for the purpose of illustration, we come back to the tracking configuration as in Figure 17 with a plant model G_Δ. If we employ the representation in Figure 38, we arrive at Figure 39.

How do we pull Δ out of the interconnection? We simply disconnect Δ and K to arrive at Figure 40.

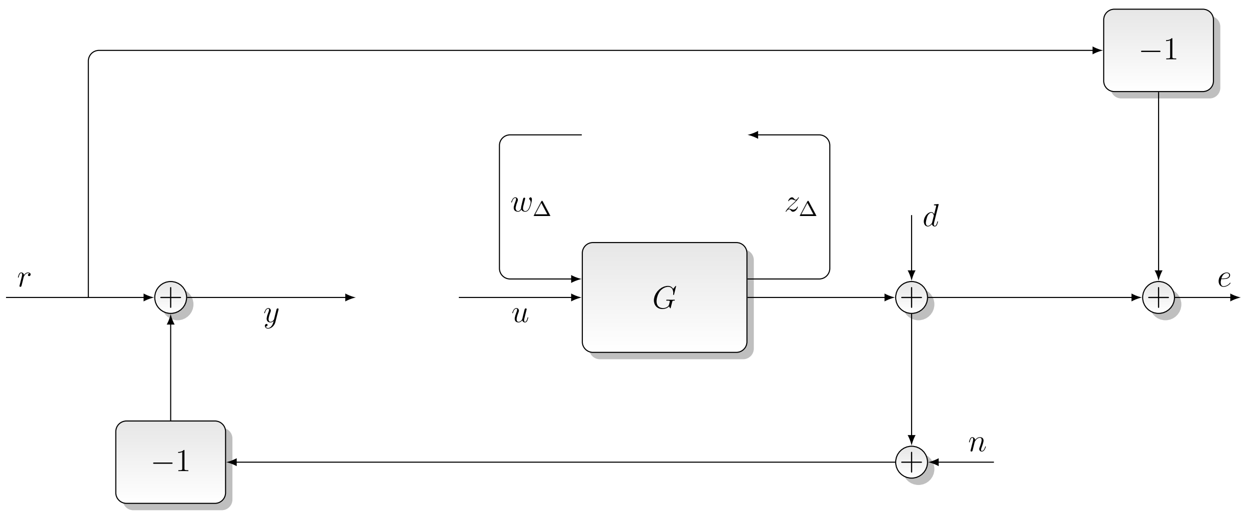


Figure 40: Uncertain open-loop interconnection

It is then not difficult to obtain the corresponding algebraic relations as

$$\begin{pmatrix} z_\Delta \\ \hline e \\ \hline y \end{pmatrix} = \left(\begin{array}{c|ccc|c} G_{11} & 0 & 0 & 0 & G_{12} \\ \hline G_{21} & I & 0 & -I & G_{22} \\ \hline G_{21} & -I & -I & I & -G_{22} \end{array}\right) \begin{pmatrix} w_\Delta \\ \hline d \\ n \\ r \\ \hline u \end{pmatrix}$$

for the open-loop interconnection. The command `sysic` is very useful in automizing the calculation of a state-space representation of this system.

After having determined this open-loop system, the uncertain uncontrolled system is obtained by re-connecting the uncertainty as

$$w_\Delta = \Delta z_\Delta.$$

Note that the uncertainty for the component is just coming back as uncertainty for the interconnection. Hence the structure for the interconnection is simply inherited.

This is different if several components of the system are affected by uncertainties that are to be pulled out. Then the various uncertainties for the components will appear on the diagonal of an uncertainty block for the interconnection.

Let us again look at an illustrative example. Suppose we would like to connect a SISO controller K to a real system. Since K has to be simulated (in a computer), the actually implemented controller will differ from K. Such a variation can be captured in an uncertainty description: the implemented controller is

$$K(I + \Delta_K)$$

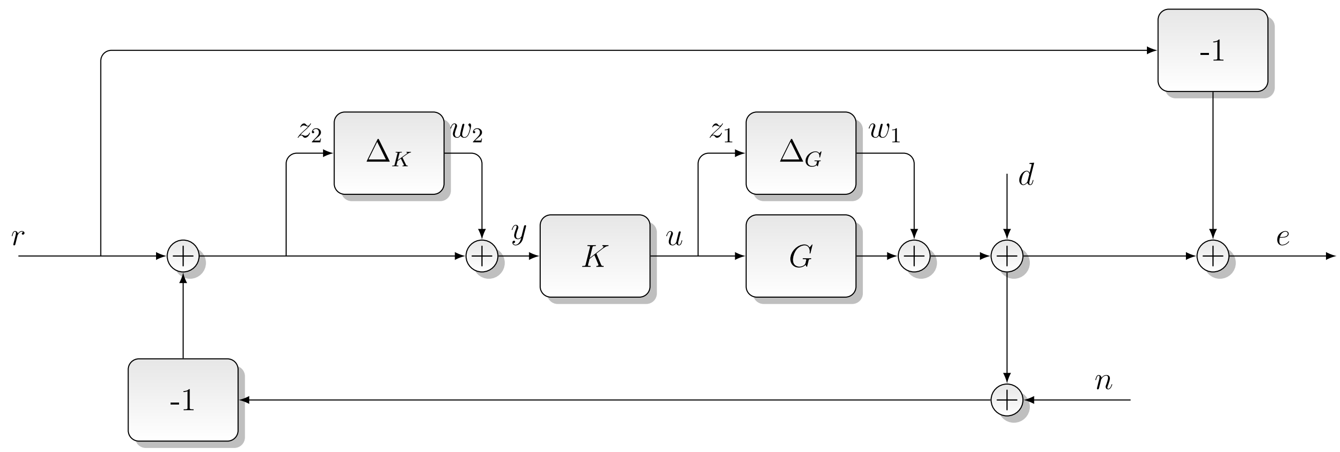


Figure 41: Uncertain closed-loop interconnection

where Δ_K is a proper and stable transfer matrix in some class that captures our knowledge of the accuracy of the implemented controller, very similar to what we have been discussing for a model of the plant.

Let us hence replace K by $K(I + \Delta_K)$ in the interconnection in Figure 25. Since this is a multiplicative input uncertainty, we arrive at Figure 41.

If we disconnect K, Δ_K, Δ_G, the resulting open-loop interconnection is given as

$$\begin{pmatrix} z_1 \\ z_2 \\ \hline e \\ u \\ \hline y \end{pmatrix} = \left(\begin{array}{cc|ccc|c} 0 & 0 & 0 & 0 & 0 & I \\ -I & 0 & -I & -I & I & G \\ \hline I & 0 & I & 0 & -I & G \\ 0 & 0 & 0 & 0 & 0 & I \\ \hline -I & I & -I & -I & I & G \end{array}\right) \begin{pmatrix} w_1 \\ w_2 \\ \hline d \\ n \\ r \\ \hline u \end{pmatrix},$$

and the perturbation enters as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \Delta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \Delta_G & 0 \\ 0 & \Delta_K \end{pmatrix}.$$

Since the signals z_1, z_2 and the signals w_1, w_2 are *different*, Δ admits a block-diagonal structure with Δ_G and Δ_K appearing on its diagonal.

Remark. Suppose that two uncertainties Δ_1, Δ_2 enter an interconnection as in Figure 42. Then they can be pulled out as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} z$$

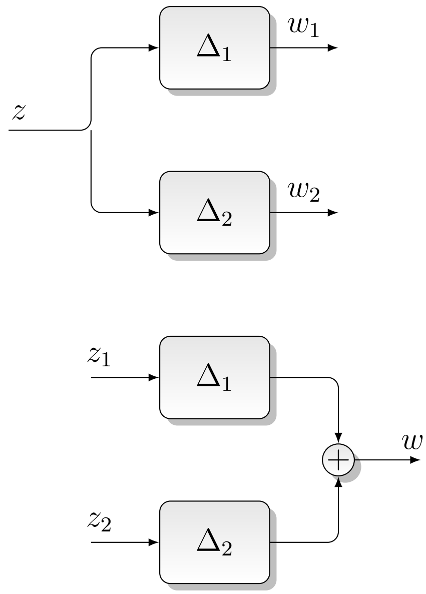


Figure 42: Special Configurations

or as

$$w = \begin{pmatrix} \Delta_1 & \Delta_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

respectively. Instead, however, it is also possible to pull them out as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

by simply neglecting the fact that Δ_1 and Δ_2 are entered by the same signal, or that the outputs sum up to one signal.

In summary, uncertainties might be structured or not at the component level. If pulling them out of an interconnection, the resulting uncertainty for the interconnection is block-diagonal, and the uncertainties of the components appear, possibly repeated, on the diagonal.

If pulling uncertainties out of an interconnection, they will automatically have a block-diagonal structure, even if the component uncertainties are not structured themselves.

4.5 The General Paradigm

We have seen how to describe a possibly complicated interconnection in the form

$$\begin{pmatrix} z_\Delta \\ z \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix}. \tag{4.19}$$

Here w_Δ, z_Δ are the signals that are introduced to pull out the uncertainties, w, z are generalized disturbance and controlled variable, and u, y are control input and measured output respectively.

The uncertainties will belong to a set $\mathbf{\Delta}$ that consists of proper and stable transfer matrices. The perturbed uncontrolled interconnection is obtained by re-connecting the uncertainty as

$$w_\Delta = \Delta z_\Delta \ \text{ with } \ \Delta \in \mathbf{\Delta}.$$

This leads to

$$\begin{aligned} \begin{pmatrix} z \\ y \end{pmatrix} &= S(\Delta, P) \begin{pmatrix} w \\ u \end{pmatrix} = \\ &= \left(\begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix} + \begin{pmatrix} P_{21} \\ P_{31} \end{pmatrix} \Delta (I - P_{11}\Delta)^{-1} \begin{pmatrix} P_{12} & P_{13} \end{pmatrix} \right) \begin{pmatrix} w \\ u \end{pmatrix}. \end{aligned} \tag{4.20}$$

If we connect the controller to the unperturbed open-loop interconnection as

$$y = Ku,$$

we obtain

$$\begin{aligned} \begin{pmatrix} z_\Delta \\ z \end{pmatrix} &= P \star K \begin{pmatrix} w_\Delta \\ w \end{pmatrix} = \\ &= \left(\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} + \begin{pmatrix} P_{13} \\ P_{23} \end{pmatrix} K (I - P_{33}K)^{-1} \begin{pmatrix} P_{31} & P_{32} \end{pmatrix} \right) \begin{pmatrix} w_\Delta \\ w \end{pmatrix} v. \end{aligned}$$

The controlled and perturbed interconnection is obtained through

$$w_\Delta = \Delta z_\Delta \ \text{ and } \ u = Ky.$$

It does not matter in which order we reconnect Δ or K. This reveals a nice property of linear fractional transformations:

$$S(\Delta, P \star K) = S(S(\Delta, P), K).$$

Hence the closed loop system admits the descriptions

$$z = S(\Delta, P \star K)w = S(S(\Delta, P), K)w.$$

So far we were sloppy in not worrying about inverses that occur in calculating star products or about any other technicalities. Let us now get rigorous and include the exact hypotheses required in the general theory. All our technical results are subject to these assumption. Hence they need to be verified before any of the presented results can be applied.

Hypothesis 4.3

- *P is a generalized plant: there exists a controller $u = Ky$ that stabilizes (4.19) in the sense of Definition 3.2.*
- *The set of uncertainties is given as*

$$\mathbf{\Delta} := \{\Delta(s) \in RH_\infty \mid \Delta(i\omega) \in \mathbf{\Delta}_c \ \textit{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}\} \tag{4.21}$$

  where $\mathbf{\Delta}_c$ is a value set of complex matrices (motivating the index c for complex) that defines the structure and the size of the uncertainties. This set $\mathbf{\Delta}_c$ has to be star-shaped with center 0:

$$\Delta_c \in \mathbf{\Delta}_c \quad \Rightarrow \quad \tau\Delta_c \in \mathbf{\Delta}_c \ \textit{ for all } \ \tau \in [0,1]. \tag{4.22}$$

- *The direct feedthrough P_{11} and $\mathbf{\Delta}_c$ are such that*

$$I - P_{11}(\infty)\Delta_c \ \textit{ is non-singular for all } \ \Delta_c \in \mathbf{\Delta}_c. \tag{4.23}$$

Comments on the hypothesis

As a fundamental requirement of any controller, it should stabilize an interconnection. Hence there should at least exist a stabilizing controller, and the tests developed in Section 3 can be applied to verify this fact.

In the second hypothesis, we define the considered class of uncertainties to be all proper and stable transfer matrices that take their values along the imaginary axis in the set $\mathbf{\Delta}_c$. We recall the specific examples we have seen earlier to illustrate this concept. It is very important to obey that this value has to be star-shapedness with center 0: If Δ_c is contained in $\mathbf{\Delta}_c$, then the whole line $\tau\Delta_c$, $\tau \in [0,1]$, that connects Δ_c with 0 belongs to $\mathbf{\Delta}_c$ as well. Note that this implies $0 \in \mathbf{\Delta}_c$ such that the zero transfer matrix is always in the class $\mathbf{\Delta}$; this is consistent with Δ_c to be viewed as a deviation from a nominal value. For sets of complex numbers, Figure 43 shows an example and a counterexample for star-shapedness.

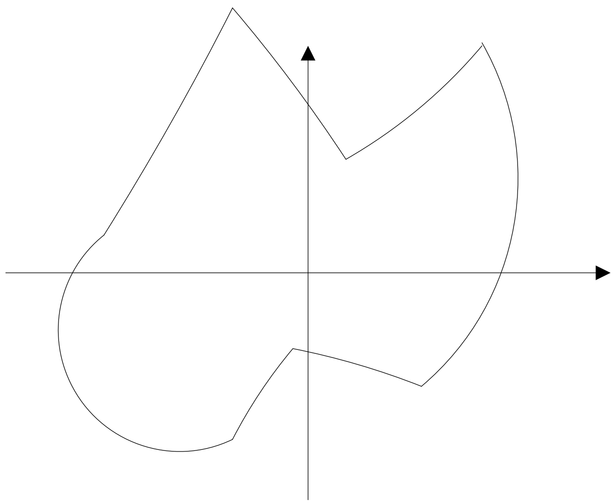


Figure 43: Star-shaped with center 0? Left: Yes. Right: No.

Remark. If $\mathbf{\Delta}_c$ is a set of matrices whose elements are just supposed to be contained in real intervals or in circles around 0 (for individual elements) or in a unit ball of matrices (for sub-blocks), the hypothesis (4.22) of star-shapedness is automatically satisfied.

The last property implies that, for any Δ in our uncertainty set $\mathbf{\Delta}$, $I - P_{11}(\infty)\Delta(\infty)$ is non-singular such that $I - P_{11}\Delta$ does have a proper inverse. This is required to guarantee that $S(\Delta, P)$ can be calculated at all (existence of inverse), and that it defines a proper transfer matrix (properness of inverse). At this stage, we don't have a systematic technique to test whether (4.23) holds true or not; this will be the topic of Section 4.9.5. However, if P_{11} is strictly proper, it satisfies $P_{11}(\infty) = 0$ and (4.23) is trivially satisfied.

Comments on weightings

- Note that we assume all weightings that are required to accurately describe the uncertainty size or structure to be absorbed in P. These weightings do not need to obey any specific technical properties; they neither need to be stable nor even proper. The only requirement is to ask P being a generalized plant - this is the decisive condition to apply our results. Of course, a wrongly chosen weighting might preclude the existence of a stabilizing controller, and hence it might destroy this property; therefore, an adjustment of the weightings might be a possibility to enforce that P is a generalized plant.

- We could as well incorporate a weighting a posteriori in P. Suppose that we actually intend to work with $\widehat{\Delta}$ that is related with Δ by

$$\Delta = W_1 \widehat{\Delta} W_2$$

for real rational weightings W_1 and W_2. Then we simply replace P by $\widehat{P}$ given as

$$\widehat{P} = \begin{pmatrix} W_2P_{11}W_1 & W_2P_{12} & W_2P_{13} \\ P_{21}W_1 & P_{22} & P_{23} \\ P_{31}W_1 & P_{32} & P_{33} \end{pmatrix}$$

and proceed with $\widehat{P}$ and $\widehat{\Delta}$. (Note that this just amounts to pulling out $\widehat{\Delta}$ in $z = W_1\widehat{\Delta}W_2w$.)

Comments on larger classes of uncertainties

- We could allow for value sets $\mathbf{\Delta}_c(\omega)$ that depend on the frequency $\omega \in \mathbb{R} \cup \{\infty\}$ in order to define the uncertainty class $\mathbf{\Delta}$. Then we require $\mathbf{\Delta}_c(\omega)$ to be star-shaped with star center 0 for all $\omega \in \mathbb{R} \cup \{\infty\}$.

- We could even just stay with a general set of $\mathbf{\Delta}$ of real rational proper and stable transfer matrices that does not admit a specific description at all. We would still require that this set is star-shaped with center 0 ($\Delta \in \mathbf{\Delta}$ implies $\tau\Delta \in \mathbf{\Delta}$ for all $\tau \in [0, 1]$), and that $I - P_{22}(\infty)\Delta(\infty)$ is non-singular for all $\Delta \in \mathbf{\Delta}$.

4.6 What is Robust Stability?

We have already seen when K achieves nominal stability: K should stabilize P in the sense of Definition 3.2.

Robust Stabilization
We say that K *robustly stabilizes* $S(\Delta, P)$ *against the uncertainties* $\Delta \in \mathbf{\Delta}$ if K stabilizes the system $S(\Delta, P)$ for any uncertainty Δ taken out of the underlying class $\mathbf{\Delta}$.

Robust Stability Analysis Problem
For a given fixed controller K, test whether it robustly stabilizes $S(\Delta, P)$ against all uncertainties in $\mathbf{\Delta}$.

Robust Stability Synthesis Problem
Find a controller K that robustly stabilizes $S(\Delta, P)$ against all uncertainties in $\mathbf{\Delta}$.

Although these definitions seem as tautologies, it is important to read them carefully: If we have not specified a set of uncertainty, it does not make sense to talk of a robustly stabilizing controller. Hence we explicitly included in the definition that robust stability is related to a well-specified set of uncertainties. Clearly, whether or not a controller robustly stabilizes an uncertain system, highly depends on the class of uncertainties that is considered. These remarks are particularly important for controller design: If one has found a controller that robustly stabilizes an uncertain system for a specific uncertainty class, there is, in general, no guarantee whatsoever that such a controller is robustly stabilizing for some other uncertainty class.[1]

4.7 Robust Stability Analysis

4.7.1 Simplify Structure

Starting from our general paradigm, we claim and prove that robust stability can be decided on the basis of the transfer matrix that is seen by Δ. Let us hence introduce the abbreviation

$$P \star K = N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

where we highlight M; this is the block referred to as the transfer matrix seen by the uncertainty.

Theorem 4.4 *If K stabilizes P, and if*

$$I - M\Delta \;\; \textit{has a proper and stable inverse for all } \Delta \in \mathbf{\Delta},$$

then K robustly stabilizes $S(\Delta, P)$ against $\mathbf{\Delta}$.

Recall that $I - M(s)\Delta(s)$ has a proper and stable inverse if and only if $I - M(s)\Delta(s)$ is non-singular for all $s \in \mathbb{C}_= \cup \mathbb{C}_> \cup \{\infty\}$ or, equivalently,

$$\det(I - M(s)\Delta(s)) \neq 0 \;\; \text{for all} \;\; s \in \mathbb{C}_= \cup \mathbb{C}_> \cup \{\infty\}.$$

Note that it is difficult to verify the latter property for all $\Delta \in \mathbf{\Delta}$. The crux of this result is a structural simplification: Instead of investigating $S(\Delta, P)$ that depends possibly in a highly involved fashion on Δ, we only need to investigate $I - M\Delta$ which is just linear in Δ. Hence, it is sufficient to look at this generic structure for all possible (potentially complicated) interconnections.

[1]These remarks are included since, recently, it has been claimed in a publication that robust control design techniques are not useful, based on the following argument: The authors of this paper constructed a controller that robustly stabilizes a system for some uncertainty class, and they tested this controller against another class. It turned out that the robustness margins (to be defined later) for the alternative uncertainty class are poor. The authors conclude that the technique they employed is not suited to design robust controllers. This is extremely misleading since the actual conclusion should read as follows: There is no reason to expect that a robustly stabilizing controller for one class of uncertainties is also robustly stabilizing for another (possibly unrelated) class of uncertainties. Again, this is logical and seems almost tautological, but we stress these points since the earlier mentioned severe confusions arose in the literature.

Proof. Taking any $\Delta \in \mathbf{\Delta}$, we have to show that $u = Ky$ stabilizes the system in (4.20) in the sense of Definition 3.2. At this point we benefit from the fact that we don't need to go back to the original definition, but, instead, we can argue in terms of input-output descriptions as formulated in Theorem 3.5. We hence have to show that

$$\begin{pmatrix} z \\ y \end{pmatrix} = S(\Delta, P) \begin{pmatrix} w \\ u \end{pmatrix}, \quad u = Kv + v_1, \;\; v = y + v_2 \tag{4.24}$$

defines a proper and stable system $\begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} z \\ u \\ v \end{pmatrix}$. Clearly, we can re-represent this system as

$$\begin{pmatrix} z_\Delta \\ z \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix}, \quad u = Kv + v_1, \;\; v = y + v_2, \quad w_\Delta = \Delta z_\Delta. \tag{4.25}$$

Recall that K stabilizes P. Hence the relations

$$\begin{pmatrix} z_\Delta \\ z \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix}, \quad u = Kv + v_1, \;\; v = y + v_2,$$

define a stable LTI system

$$\left(\begin{array}{c} z_\Delta \\ z \\ \hline u \\ v \end{array}\right) = \left(\begin{array}{cc|cc} M & N_{12} & H_{13} & H_{14} \\ N_{21} & N_{22} & H_{23} & H_{24} \\ \hline H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{array}\right) \left(\begin{array}{c} w_\Delta \\ w \\ \hline v_1 \\ v_2 \end{array}\right). \tag{4.26}$$

In addition to $N = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$, several other blocks appear in this representation whose structure is not important; the only important fact is that they are all proper and stable.

If we reconnect $w_\Delta = \Delta z_\Delta$ in (4.26), we arrive at an alternative representation of (4.25) or of (4.24) that reads as

$$\begin{pmatrix} z \\ u \\ v \end{pmatrix} = \left(\begin{pmatrix} N_{22} & H_{23} & H_{24} \\ H_{32} & H_{33} & H_{34} \\ H_{42} & H_{43} & H_{44} \end{pmatrix} + \begin{pmatrix} N_{21} \\ H_{31} \\ H_{41} \end{pmatrix} \Delta (I - M\Delta)^{-1} \begin{pmatrix} N_{12} & H_{13} & H_{14} \end{pmatrix} \right) \begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix}.$$

The essential point: Since both Δ and $(I - M\Delta)^{-1}$ are proper and stable, and since, as mentioned above, all the other blocks occurring in this formula are proper and stable as well, this system defines a proper and stable transfer matrix as we had to prove.

■

4.7.2 Reduction to Non-Singularity Test on Imaginary Axis

Recall that we need to verify whether $I - M\Delta$ has a proper and stable inverse; for that purpose one has to check whether the matrix $I - M\Delta$ itself does not have zeros in the closed right-half plane including infinity. Hence we need to check

$$\det(I - M(s)\Delta(s)) \neq 0 \ \text{ for all } \ s \in \mathbb{C}_= \cup \mathbb{C}_> \cup \{\infty\}, \ \Delta \in \boldsymbol{\Delta}. \tag{4.27}$$

This is complicated since we have to scan the full right-half plane, and we have to perform the test for all dynamic uncertainties under consideration.

The following result shows that it suffices to test $I - M(s)\Delta_c$ for non-singularity only for $s = i\omega$ with $\omega \in \mathbb{R} \cup \{\infty\}$, and only for $\Delta_c \in \boldsymbol{\Delta}_c$. Hence this reduces the original problem to a pure problem in linear algebra, what might considerably simplify the test.

Let us first formulate the precise result.

Theorem 4.5 *Suppose M is a proper and stable transfer matrix. If*

$$\det(I - M(i\omega)\Delta_c) \neq 0 \ \textit{ for all } \ \Delta_c \in \boldsymbol{\Delta}_c, \ \omega \in \mathbb{R} \cup \{\infty\}, \tag{4.28}$$

then

$$I - M\Delta \ \textit{ has a proper and stable inverse for all } \Delta \in \boldsymbol{\Delta}. \tag{4.29}$$

Before we provide a formal proof, we would like to provide some intuition why the star-shapedness hypothesis plays an important role in this result. Let us hence assume that (4.28) is valid. Obviously, we can then conclude that

$$\det(I - M(\lambda)\Delta(\lambda)) \neq 0 \ \text{ for all } \ \lambda \in \mathbb{C}_= \cup \{\infty\}, \ \Delta \in \boldsymbol{\Delta}, \tag{4.30}$$

since $\Delta(\lambda)$ is contained in $\boldsymbol{\Delta}_c$ if $\lambda \in \mathbb{C}_= \cup \{\infty\}$ and $\Delta \in \boldsymbol{\Delta}$. Note that (4.27) and (4.30) just differ by replacing $\mathbb{C}_= \cup \mathbb{C}_> \cup \{\infty\}$ with $\mathbb{C}_= \cup \{\infty\}$. Due to $\mathbb{C}_= \cup \{\infty\} \subset \mathbb{C}_= \cup \mathbb{C}_> \cup \{\infty\}$, it is clear that (4.27) implies (4.30). However, we need the converse: We want to conclude that (4.30) implies (4.27), and this is the non-trivial part of the story.

Why does this implication hold? We have illustrated the following discussion in Figure 44. The proof is by contradiction: Assume that (4.27) is not valid, and that (4.30) is true. Then there exists a $\Delta \in \boldsymbol{\Delta}$ for which $I - M\Delta$ has a zero s_1 in $\mathbb{C}_>$ (due to (4.27)) where

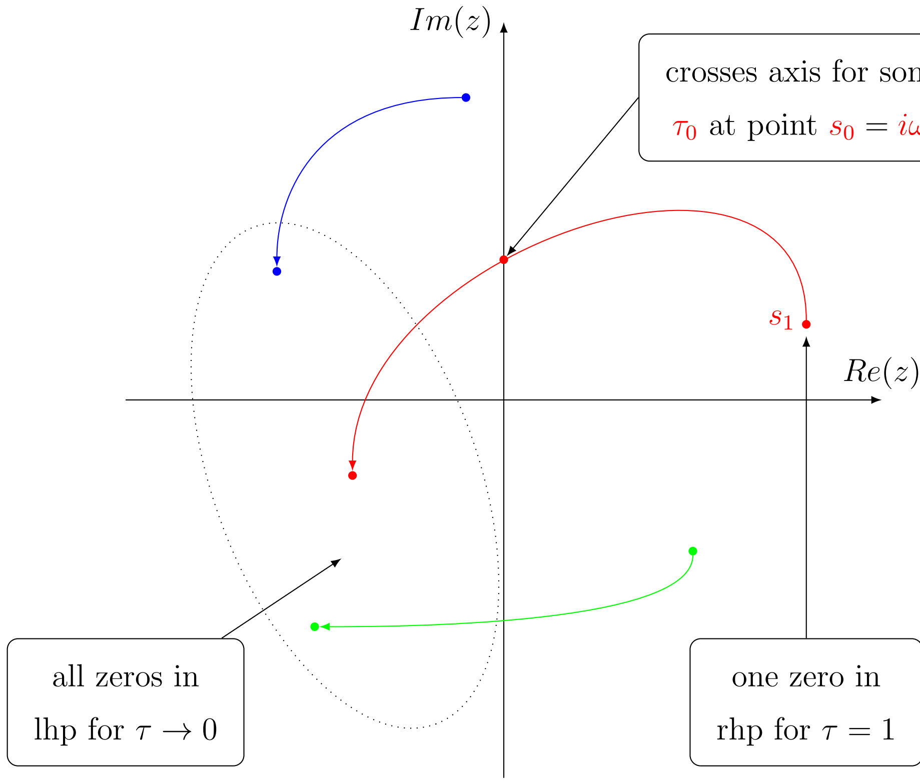


Figure 44: Movements of zeros

s_1 is certainly not contained in $\mathbb{C}_= \cup \{\infty\}$ (due to (4.30)). For this single Δ, we cannot detect that it is destabilizing without scanning the right-half plane. However, apart from Δ, we can look at all the uncertainties $\tau\Delta$ obtained by varying $\tau \in [0, 1]$. Since $\mathbf{\Delta}_c$ is star-shaped, all these uncertainties are contained in the set $\mathbf{\Delta}$ as well. (Check that!) Let us now see what happens to the zeros of

$$\det(I - M(s)[\tau\Delta(s)]).$$

For $\tau = 1$, this function has the zero s_1 in $\mathbb{C}_>$. For τ close to zero, one can show that all its zeros must be contained in $\mathbb{C}_<$. (The loop is stable for $\tau = 0$ such that it remains stable for τ close to zero since, then, the perturbation $\tau\Delta$ is small as well; we provide a proof that avoids these sloppy reasonings.) Therefore, if we let τ decrease from 1 to 0, we can expect that the unstable zero s_1 has to move from $\mathbb{C}_>$ to $\mathbb{C}_<$. Since it moves continuously, it must hit the imaginary axis on its way for some parameter τ_0.[2] If this zero curve hits the imaginary axis at $i\omega_0$, we can conclude that

$$\det(I - M(i\omega_0)[\tau_0\Delta(i\omega_0)]) = 0.$$

Hence $\tau_0\Delta(s)$ is an uncertainty that is still contained in $\mathbf{\Delta}$ (star-shapeness!) and for which $I - M[\tau_0\Delta]$ has a zero at $i\omega_0$. We have arrive at the contradiction that (4.30) cannot be true either.

[2]Contrary to what is often stated in the literature, this is *not* an elementary continuity argument!

It is interesting to summarize what these arguments reveal: If we find a Δ such that $(I - M\Delta)^{-1}$ has a pole in the open right-half plane, then we can also find another $\tilde{\Delta}$ for which $(I - M\tilde{\Delta})^{-1}$ has a pole on the imaginary axis.

Comments on larger classes of uncertainties

- If $\mathbf{\Delta}_c(\omega)$ depends on frequency (and is star-shaped), we just have to check

$$\det(I - M(i\omega)\Delta_c) \neq 0 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}, \ \Delta_c \in \mathbf{\Delta}_c(\omega)$$

  in order to conclude (4.29). The proof remains unchanged.

- If $\mathbf{\Delta}$ is a general set of real rational proper and stable transfer matrices without specific description, we have to directly verify (4.30) in order to conclude (4.29).

Let us now provide the rigorous arguments to finish the proof of Theorem 4.5.

Proof. Recall that it remains to show (4.30) $\Rightarrow$ (4.27) by contradiction. Suppose (4.30) holds, but (4.27) is not valid. Hence there exists a $s_1 \in \mathbb{C}_>$, $s_1 \notin \mathbb{C}_= \cup \{\infty\}$, and a $\Delta_1 \in \mathbf{\Delta}$ such that

$$\det(I - M(s_1)\Delta_1(s_1)) = 0. \tag{4.31}$$

If we can show that there exists a $s_0 \in \mathbb{C}_=$ and a $\tau_0 \in [0,1]$ for which

$$\det(I - M(s_0)[\tau_0\Delta_1(s_0)]) = 0, \tag{4.32}$$

we arrive at a contradiction to (4.30) since $\tau_0\Delta \in \mathbf{\Delta}$ and $s_0 \in \mathbb{C}_=$.

To find s_0 and τ_0, let us take the realization

$$M\Delta_1 = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

We obviously have

$$I - M[\tau\Delta_1] = \left[\begin{array}{c|c} A & B \\ \hline -\tau C & I - \tau D \end{array}\right]$$

and the well-known Schur formula leads to

$$\det(I - M(s)[\tau\Delta(s)]) = \frac{\det(I - \tau D)}{\det(sI - A)} \det(sI - A(\tau)) \tag{4.33}$$

if we abbreviate

$$A(\tau) = A + B(I - \tau D)^{-1}\tau C.$$

If we apply (4.30) for $s = \infty$ and $\Delta = \tau\Delta_1$, we infer that $\det(I - \tau D) \neq 0$ for all $\tau \in [0,1]$. In addition, A is stable such that $\det(s_1 I - A) \neq 0$. If we hence combine (4.33) and (4.31), we arrive at

$$\det(s_1 I - A(1)) = 0 \ \text{ or } \ s_1 \in \lambda(A(1)).$$

Let us now exploit as a fundamental result the continuous dependence of eigenvalues of matrices: Since $A(\tau)$ depends continuously on $\tau \in [0,1]$, there exists a continuous function $s(.)$ defined on $[0,1]$ and taking values in the complex plane such that

$$s(1) = s_1, \quad \det(s(\tau)I - A(\tau)) = 0 \;\text{ for all }\; \tau \in [0,1].$$

($s(.)$ defines a continuous curve in the complex plane that starts in s_1 and such that, for each τ, $s(\tau)$ is an eigenvalue of $A(\tau)$.) Now we observe that $A(0) = A$ is stable. Therefore, $s(0)$ must be contained in $\mathbb{C}_<$. We conclude: the continuous function $\mathrm{Re}(s(\tau))$ satisfies

$$\mathrm{Re}(s(0)) < 0 \;\text{ and }\; \mathrm{Re}(s(1)) > 0.$$

Hence there must exist a $\tau_0 \in (0,1)$ with

$$\mathrm{Re}(s(\tau_0)) = 0.$$

Then $s_0 = s(\tau_0)$ and τ_0 lead to (4.32) what is the desired contradiction.

■

For later purposes we can modify the proof to obtain the following result.

Lemma 4.6 *Let P be stable with $\|P\|_\infty \leq 1$. Suppose K stabilizes P, that K has no poles in $\mathbb{C}^0$ and satisfies $\|K(i\omega)\| < 1$ for all $\omega \in [0,\infty]$. Then K is stable.*

Proof. Choose $\omega \in [0,\infty]$ and $\tau \in [0,1]$ arbitrary. Then we have

$$\|\tau P(i\omega)K(i\omega)\| \leq P(i\omega)\| \; \|K(i\omega)\| < 1.$$

This implies

$$\det(I - \tau P(i\omega)K(i\omega)) \neq 0.$$

Choose again minimal realizations (A_p, B_p, C_p, D_p) and (A_K, B_K, C_K, D_K) of P and K and abbreviate

$$PK = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{cc|c} A_p & B_pC_K & B_pD_K \\ 0 & A_K & B_K \\ \hline C_p & D_pC_K & D_pD_K \end{array}\right].$$

Define again $A(\tau) := A + B(I - \tau D)^{-1}\tau C$ to obtain with the Schur formula

$$\det(I - P(s)[\tau K(s)]) = \frac{\det(I - \tau D)}{\det(sI - A)} \det(sI - A(\tau)).$$

Set now $S_i := (I - D_K D_p)^{-1}$ and $S_o := (I - D_p D_K)^{-1}$ then observe that

$$\begin{aligned}
A(1) &= A + BS_oC \\
&= \begin{pmatrix} A_p & B_pC_K \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_pD_K \\ B_K \end{pmatrix} S_o \begin{pmatrix} C_p & D_pC_K \end{pmatrix} \\
&= \begin{pmatrix} A_p & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_pS_iD_KC_p & B_pS_iC_K \\ B_KS_oC_p & B_KS_oD_pC_K \end{pmatrix} \\
&= \begin{pmatrix} A_p & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_p & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} S_i & S_iD_K \\ S_oD_p & S_o \end{pmatrix} \begin{pmatrix} 0 & C_K \\ C_p & 0 \end{pmatrix} \\
&= \begin{pmatrix} A_p & 0 \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_p & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} I & -D_K \\ -D_p & I \end{pmatrix}^{-1} \begin{pmatrix} 0 & C_K \\ C_p & 0 \end{pmatrix}
\end{aligned}$$

is Hurwitz since K stabilizes P. With the same idea as in the proof of Theorem 4.5 we can conclude that $A(0) = \begin{pmatrix} A_p & B_pC_K \\ 0 & A_K \end{pmatrix}$ is Hurwitz and hence A_K is Hurwitz. This means that K is stable.

■

4.7.3 The Central Test for Robust Stability

We can easily combine Theorem 4.4 with Theorem 4.5 to arrive at the fundamental robust stability analysis test for controlled interconnections.

Corollary 4.7 *If K stabilizes P, and if*

$$\det(I - M(i\omega)\Delta_c) \neq 0 \ \ \textit{for all} \ \ \Delta_c \in \mathbf{\Delta}_c, \ \omega \in \mathbb{R} \cup \{\infty\}, \tag{4.34}$$

then K robustly stabilizes $S(\Delta, P)$ against $\mathbf{\Delta}$.

Contrary to what is often claimed in the literature, the converse does in general not hold in this result. Hence (4.34) might in general *not be a tight condition*. In practice and for almost all relevant uncertainty classes, however, it often turns out to be tight. In order to show that the condition is tight for a specific setup, one can simply proceed as follows: if (4.34) is not true, try to construct a *destabilizing perturbation*, an uncertainty $\Delta \in \mathbf{\Delta}$ for which K does not stabilize $S(\Delta, P)$.

The construction of destabilizing perturbations is the topic of the next section.

4.7.4 Construction of Destabilizing Perturbations

As already pointed out, this section is related to the question in how far condition (4.34) in Theorem 4.7 is also *necessary* for robust stability. We are not aware of definite answers to this questions in our general setup, but we are aware of some false statements in the literature! Nevertheless, we do not want to get into a technical discussion but we intend to take a pragmatic route in order to construct destabilizing perturbations.

Let us assume that (4.34) is not valid. This means that we can find a complex matrix $\Delta_0 \in \mathbf{\Delta}_c$ and a frequency $\omega_0 \in \mathbb{R} \cup \{\infty\}$ for which

$$I - M(i\omega_0)\Delta_0 \ \text{ is singular.}$$

First step in constructing a destabilizing perturbation
Find a real rational proper and stable $\Delta(s)$ with

$$\Delta(i\omega_0) = \Delta_0 \ \text{ and } \ \Delta(i\omega) \in \mathbf{\Delta}_c \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}. \tag{4.35}$$

This implies that Δ is contained in our class $\mathbf{\Delta}$, and that $I - M\Delta$ does not have a proper and stable inverse (since $I - M(i\omega_0)\Delta(i\omega_0) = I - M(i\omega_0)\Delta_0$ is singular.)

Comments
Note that the construction of Δ amounts to solving an interpolation problem: The function $\Delta(s)$ should be contained in our class $\mathbf{\Delta}$ and it should take the value Δ_0 at the point $s = i\omega_0$.

This problem has a *trivial* solution if Δ_0 is a *real* matrix. Just set

$$\Delta(s) := \Delta_0.$$

If Δ_0 is *complex*, this choice is not suited since it is not contained in our perturbation class of *real* rational transfer matrices. In this case we need to do some work. Note that this was the whole purpose of Lemma **??** if Δ_c is the open unit disk in the complex plane. For more complicated sets (such as for block diagonal structures) we comment on the solution of this problem later.

Second step in constructing a destabilizing perturbation
For the constructed Δ, check whether K stabilizes $S(\Delta, P)$ by any of our tests developed earlier. If the answer is no, we have found a destabilizing perturbation. If the answer is yes, the question of whether K is robustly stabilizing remains undecided.

Comments
In most practical cases, the answer will be no and the constructed Δ is indeed destabilizing. However, one can find examples where this is not the case, and this point is largely overlooked in the literature.

Let us provide (without proofs) conditions under which *we can be sure that Δ is destabilizing*:

- If $\omega_0 = \infty$ then Δ is destabilizing.
- If ω_0 is finite, if
$$\left(\begin{array}{c|cc|c} A - i\omega_0 I & 0 & B_1 & B_2 \\ \hline 0 & \Delta_0 & -I & 0 \\ C_1 & -I & D_{11} & D_{12} \end{array}\right) \quad \text{has full row rank,}$$
and if
$$\left(\begin{array}{c|cc} A - i\omega_0 I & 0 & B_1 \\ \hline 0 & \Delta_0 & -I \\ C_1 & -I & D_{11} \\ \hline C_2 & 0 & D_{21} \end{array}\right) \quad \text{has full column rank,}$$
then Δ is destabilizing.
- This latter test can be re-formulated in terms of P. If ω_0 is finite, if $i\omega_0$ is not a pole of $P(s)$, if
$$\left(I - P_{11}(i\omega_0)\Delta_0 \;\; P_{12}(i\omega) \right) \quad \text{has full row rank,}$$
and if
$$\begin{pmatrix} I - \Delta_0 P_{11}(i\omega_0) \\ P_{21}(i\omega_0) \end{pmatrix} \quad \text{has full column rank,}$$
then Δ is destabilizing.

Note that these conditions are very easy to check, and they will be true in most practical cases. If they are not valid, the question of robust stability remains undecided. The failing of the latter conditions might indicate that the process of pulling out the uncertainties can be performed in a more efficient fashion by reducing the size of Δ.

4.8 Important Specific Robust Stability Tests

Testing (4.34) can be still pretty difficult in general, since the determinant is a complicated function of the matrix elements, and since we still have to test an infinite number of matrices for non-singularity.

On the other hand, very many robust stability test for LTI uncertainties have (4.34) at their roots, and this condition can be specialized to simple tests in various interesting settings. We can only touch upon the wealth of consequences.

4.8.1 M is Scalar

If it happens that $M(i\omega)$ has dimension 1×1, then $\mathbf{\Delta}_c$ is simply a set of complex numbers. In this case (4.34) just amounts to testing $1-M(i\omega)\Delta_c \neq 0$ for all $\Delta_c \in \mathbf{\Delta}_c$. This amounts to testing, frequency by frequency, whether the set

$$M(i\omega)\boldsymbol{\Delta_c}$$

contains 1 or not. If no, condition (4.34) holds true and we conclude robust stability. If 1 is contained in this set for some frequency, (4.34) fails, and we might construct a destabilizing uncertainty as in Section 4.7.4.

In many cases, $\mathbf{\Delta}_c$ is the open unit circle in the complex plane. If $1 \in M(i\omega_0)\Delta_0$, Lemma **??** allows us to construct a proper real rational stable $\Delta(s)$ with $\Delta(i\omega_0) = \Delta_0$. This is the candidate for a destabilizing perturbation as discussed in Section 4.7.4.

4.8.2 The Small-Gain Theorem

Let us be specific and assume that the dimension of the uncertainty block is $p \times q$. Then we infer

$$\mathbf{\Delta}_c \subset \mathbb{C}^{p\times q}.$$

No matter whether or not $\mathbf{\Delta}_c$ consists of structure or unstructured matrices, it will certainly be a bounded set. Let us assume that we have found an r for which any

$$\Delta_c \in \mathbf{\Delta}_c \;\text{ satisfies }\; \|\Delta_c\| < r.$$

In order to check (4.34), we choose an arbitrary $\omega \in \mathbb{R}\cup\{\infty\}$, and any $\Delta_c \in \mathbf{\Delta}_c$.

We infer that

$$\det(I - M(i\omega)\Delta_c) \neq 0 \tag{4.36}$$

is equivalent, by the definition of eigenvalues, to

$$1 \notin \lambda(M(i\omega)\Delta_c). \tag{4.37}$$

1 is certainly not an eigenvalue of $M(i\omega)\Delta_c$ if all these eigenvalues are in absolute value smaller than 1. Hence (4.37) follows from

$$\rho(M(i\omega)\Delta_c) < 1. \tag{4.38}$$

Since the spectral radius is smaller than the spectral norm of $M(i\omega)\Delta_c$, (4.38) is implied by

$$\|M(i\omega)\Delta_c\| < 1. \tag{4.39}$$

Finally, since the norm is sub-multiplicative, (4.39) follows from

$$\|M(i\omega)\|\|\Delta_c\| < 1. \tag{4.40}$$

At this point we exploit our knowledge that $\|\Delta_c\| < r$ to see that (4.40) is a consequence of

$$\|M(i\omega)\| \le \frac{1}{r}. \tag{4.41}$$

We have seen that all the properties (4.37)-(4.41) are *sufficient conditions* for (4.34) and, hence, for robust stability.

Why have we listed all these conditions in such a detail? They all appear - usually separately - in the literature under the topic 'small gain'. However, these conditions are not often related to each other such that it might be very confusing what the right choice is. The above chain of implications gives the answer: They all provide sufficient conditions for (4.36) to hold.

Recall that we have to guarantee (4.36) for all $\omega \in \mathbb{R} \cup \{\infty\}$. In fact, this is implied if (4.41) holds for all $\omega \in \mathbb{R} \cup \{\infty\}$, what is in turn easily expressed as $\|M\|_\infty \le \frac{1}{r}$.

Theorem 4.8 *If any $\Delta_c \in \mathbf{\Delta}_c$ satisfies $\|\Delta_c\| < r$, and if*

$$\|M\|_\infty \le \frac{1}{r}, \tag{4.42}$$

then $I - M\Delta$ has a proper and stable inverse for all $\Delta \in \mathbf{\Delta}$.

Again, one can combine Theorem 4.8 with Theorem 4.4 to see that (4.42) is a sufficient condition for K to robustly stabilize $S(\Delta, P)$.

Corollary 4.9 *If K stabilizes P, and if*

$$\|M\|_\infty \le \frac{1}{r},$$

then K robustly stabilizes $S(\Delta, P)$ against $\mathbf{\Delta}$.

We stress again that (4.42) is only *sufficient*; it neglects any structure that might be characterized through $\mathbf{\Delta}_c$, and it only exploits that all the elements of this set are bounded by r.

Remarks. Note that this result also holds for an *arbitrary* class $\mathbf{\Delta}$ of real rational proper and stable matrices (no matter how they are defined) if they all satisfy the bound

$$\|\Delta\|_\infty < r.$$

Moreover, we are not at all bound to the specific choice of $\|.\| = \sigma_{\max}(.)$ as a measure for the size of the underlying complex matrices. We could replace (also in the definition of $\|.\|_\infty$) the maximal singular value by an *arbitrary* norm on matrices that is induced by vector norms, and all our results remain valid. This would lead to another bunch of small gain theorems that lead to *different* conditions. As specific examples, think of the maximal row sum or maximal column sum which are both induced matrix norms.

4.8.3 Full Block Uncertainties

Let us suppose we have an interconnection in which only one subsystem is subject to unstructured uncertainties. If this subsystem is SISO system, we can pull out the uncertainty of the interconnection and we end up with an uncertainty block Δ of dimension 1×1. If the subsystem is MIMO, the block will be matrix valued. Let us suppose that the dimension of this block is $p \times q$, and that is only restricted in size and bounded by r.

In our general scenario, this amounts to

$$\mathbf{\Delta}_c = \{\Delta \in \mathbb{C}^{p\times q} \mid \|\Delta\| < r\}.$$

In other words, $\mathbf{\Delta}$ simply consists of all real rational proper and stable $\Delta(s)$ whose H_∞ norm is smaller than r:

$$\mathbf{\Delta} := \{\Delta \in RH_\infty^{p\times q} \mid \|\Delta\|_\infty < r\}. \tag{4.43}$$

Recall from Theorem 4.8: $\|M\|_\infty \leq \frac{1}{r}$ implies that $I - M\Delta$ has a proper and stable inverse for all $\Delta \in \mathbf{\Delta}$.

The whole purpose of this section is to demonstrate that, since $\mathbf{\Delta}$ consists of unstructured uncertainties, the converse holds true as well: If $I - M\Delta$ has a proper and stable inverse for all $\Delta \in \mathbf{\Delta}$, then $\|M\|_\infty \leq \frac{1}{r}$.

Theorem 4.10 *Let $\mathbf{\Delta}$ be defined by (4.43). Then $\|M\|_\infty \leq \frac{1}{r}$ holds true if and only if $I - M\Delta$ has a proper and stable inverse for all $\Delta \in \mathbf{\Delta}$.*

We can put it in yet another form: In case that

$$\|M\|_\infty > \frac{1}{r}, \tag{4.44}$$

we can *construct* - as shown in the proof - a real rational proper and stable Δ with $\|\Delta\|_\infty < r$ such that

$$I - M\Delta \;\text{ does not have a proper and stable inverse.} \tag{4.45}$$

This construction leads to a destabilizing perturbation for $(I - M\Delta)^{-1}$, and it is a candidate for a destabilizing perturbation of the closed-loop interconnection as discussed in Section 4.7.4.

Proof. This is what we have to do: If (4.44) holds true, there exists a $\Delta \in \mathbf{\Delta}$ with (4.45).

First step. Suppose that we have found $\omega_0 \in \mathbb{R} \cup \{\infty\}$ with $\|M(i\omega_0)\| > \frac{1}{r}$ (which exists by (4.44)). Recall that $\|M(i\omega_0)\|^2$ is an eigenvalue of $M(i\omega_0)M(i\omega_0)^*$. Hence there exists an eigenvector $u \neq 0$ with

$$[M(i\omega_0)M(i\omega_0)^*]u = \|M(i\omega_0)\|^2 u.$$

Let us define

$$v := \frac{1}{\|M(i\omega_0)\|^2} M(i\omega_0)^* u \quad \text{and} \quad \Delta_0 := v \frac{u^*}{\|u\|^2}.$$

(Note that Δ_0 has rank one; this is not important for our arguments.) We observe

$$\|\Delta_0\| \leq \frac{\|v\|}{\|u\|} \leq \frac{1}{\|M(i\omega_0)\|^2} \|M(i\omega_0)^* u\| \frac{1}{\|u\|} \leq \frac{1}{\|M(i\omega_0)\|} < r$$

and

$$[I - M(i\omega_0)\Delta_0]u = u - M(i\omega_0)v =$$
$$= u - \frac{1}{\|M(i\omega_0)\|^2} M(i\omega_0)M(i\omega_0)^* u = u - \frac{\|M(i\omega_0)\|^2}{\|M(i\omega_0)\|^2} u = 0.$$

We have constructed a complex matrix Δ_0 that satisfies

$$\|\Delta_0\| < r \quad \text{and} \quad \det(I - M(i\omega_0)\Delta_0) = 0.$$

Second step. Once we are at this point, we have discussed in Section 4.7.4 that it suffices to construct a real rational proper and stable $\Delta(s)$ satisfying

$$\Delta(i\omega_0) = \Delta_0 \quad \text{and} \quad \|\Delta\|_\infty < r.$$

Then this uncertainty renders $(I - M\Delta)^{-1}$ non-existent, non-proper, or unstable.

If $\omega_0 = \infty$ or $\omega_0 = 0$, $M(i\omega)$ is *real*. Then u can be chosen real such that Δ_0 is a real matrix. Obviously, $\Delta(s) := \Delta_0$ does the job.

Hence suppose $\omega_0 \in (0, \infty)$. Let us now apply Lemma **??** to each of the components of $\frac{u^*}{\|u\|^2} = \begin{pmatrix} u_1 & \cdots & u_q \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix}$: There exist $\alpha_j \geq 0$ and $\beta_j \geq 0$ with

$$u_j = \pm |u_j| \frac{i\omega_0 - \alpha_j}{i\omega_0 + \alpha_j}, \quad v_j = \pm |v_j| \frac{i\omega_0 - \beta_j}{i\omega_0 + \beta_j}.$$

Define the proper and stable

$$u(s) := \begin{pmatrix} \pm|u_1|\frac{s-\alpha_1}{s+\alpha_1} & \cdots & \pm|u_q|\frac{s-\alpha_q}{s+\alpha_q} \end{pmatrix}, \quad v(s) := \begin{pmatrix} \pm|v_1|\frac{s-\beta_1}{s+\beta_1} \\ \vdots \\ \pm|v_p|\frac{s-\beta_p}{s+\beta_p} \end{pmatrix}.$$

We claim that

$$\Delta(s) := v(s)u(s)$$

does the job. It is proper, stable, and it clearly satisfies $\Delta(i\omega_0) = v\frac{u^*}{\|u\|^2} = \Delta_0$. Finally, observe that

$$\|u(i\omega)\|^2 = \sum_{j=1}^{q} |u_j|^2 \left|\frac{i\omega - \alpha_j}{i\omega + \alpha_j}\right|^2 = \sum_{j=1}^{q} |u_j|^2 = \left\|\frac{u^*}{\|u\|^2}\right\|^2 = \frac{1}{\|u\|^2}$$

and

$$\|v(i\omega)\|^2 = \sum_{j=1}^{p} |v_j|^2 \left|\frac{i\omega - \beta_j}{i\omega + \beta_j}\right|^2 = \sum_{j=1}^{p} |v_j|^2 = \|v\|^2.$$

Hence $\|\Delta(i\omega)\| \leq \|u(i\omega)\|\|v(i\omega)\| = \frac{\|v\|}{\|u\|} < r$, what implies $\|\Delta\|_\infty < r$.

■

4.9 The Structured Singular Value in a Unifying Framework

All our specific examples could be reduced to uncertainties whose values on the imaginary axis admit the structure

$$\Delta_c = \begin{pmatrix} p_1 I & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & p_{n_r} I & & & & & & \\ & & & \delta_1 I & & & & & \\ & & & & \ddots & & & & \\ & & & & & \delta_{n_c} I & & & \\ & & & & & & \Delta_1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \Delta_{n_f} \end{pmatrix} \in \mathbb{C}^{p\times q} \tag{4.46}$$

and whose diagonal blocks satisfy

- $p_j \in \mathbb{R}$ with $|p_j| < 1$ for $j = 1, \ldots, n_r$,
- $\delta_j \in \mathbb{C}$ with $|\delta_j| < 1$ for $j = 1, \ldots, n_c$,

- $\Delta_j \in \mathbb{C}^{p_j \times q_j}$ with $\|\Delta_j\| < 1$ for $j = 1, \ldots, n_f$.

$p_j I$ is said to be a real repeated block, $\delta_j I$ is called a complex repeated block, and Δ_j is called a full (complex) block. The sizes of the identities can be different for different blocks. Real full blocks usually do not occur and are, hence, not contained in the list.

Let us denote the set of all this complex matrices as $\boldsymbol{\Delta}_c$. This set $\boldsymbol{\Delta}_c$ is very easy to describe: one just needs to fix for each diagonal block its structure (real repeated, complex repeated, complex full) and its dimension. If the dimension of $p_j I$ is r_j, and the dimension of $\delta_j I$ is c_j, the μ-tools expect a description of this set in the following way:

$$\texttt{blk} = \begin{pmatrix} -r_1 & 0 \\ \vdots & \vdots \\ -r_{n_r} & 0 \\ c_1 & 0 \\ \vdots & \vdots \\ c_{n_c} & 0 \\ p_1 & q_1 \\ \vdots & \vdots \\ p_{n_f} & q_{n_f} \end{pmatrix}.$$

Hence the row $(-r_j \quad 0)$ indicates a real repeated block of dimension r_j, whereas $(c_j \quad 0)$ corresponds to a complex repeated block of dimension c_j, and $(p_j \quad q_j)$ to a full block dimension $p_j \times q_j$.

Remark. In a practical example it might happen that the *ordering* of the blocks is different from that in (4.46). Then the commands in the μ-Toolbox can still be applied as long as the block structure matrix `blk` reflects the correct order and structure of the blocks.

Remark. If Δ_c takes the structure (4.46), the constraint on the size of the diagonal blocks can be briefly expressed as $\|\Delta_c\| < 1$. Moreover, the set $r\boldsymbol{\Delta}_c$ consists of all complex matrices Δ_c that take the structure (4.46) and whose blocks are bounded in size by r: $\|\Delta_c\| < r$. Here r is just a scaling factor that will be relevant in introducing the structured singular value.

The actual set of uncertainties $\boldsymbol{\Delta}$ is, once again, the set of all real rational proper and stable Δ whose frequency response takes it values in $\boldsymbol{\Delta}_c$:

$$\boldsymbol{\Delta} := \{\Delta(s) \in RH_\infty \mid \Delta(i\omega) \in \boldsymbol{\Delta}_c \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}\}.$$

Let us now apply the test (4.28) in Theorem 4.5. At a fixed frequency $\omega \in \mathbb{R} \cup \{\infty\}$, we have to verify whether

$$I - M(i\omega)\Delta_c \;\text{ is non-singular for all }\; \Delta_c \in \mathbf{\Delta}_c. \tag{4.47}$$

This is a pure problem of linear algebra.

4.9.1 The Structured Singular Value

Let us restate the linear algebra problem we have encountered for clarity: Given the complex matrix $M_c \in \mathbb{C}^{q\times p}$ and the (relative open) set of complex matrices $\mathbf{\Delta}_c \subset \mathbb{C}^{p\times q}$, decide whether

$$I - M_c\Delta_c \;\text{ is non-singular for all }\; \Delta_c \in \mathbf{\Delta}_c. \tag{4.48}$$

The answer to this question is yes or no.

We modify the problem a bit. In fact, let us consider the scaled set $r\mathbf{\Delta}_c$ in which we have multiplied every element of $\mathbf{\Delta}_c$ with the factor r. This stretches or shrinks the set $\mathbf{\Delta}_c$ by the factor r. Then we consider the following problem:

> Determine the largest r such that $I - M_c\Delta_c$ is non-singular for all Δ_c in the set $r\mathbf{\Delta}_c$. This largest value is denoted as r_*.

In other words, calculate

$$r_* = \sup\{r \mid \det(I - M_c\Delta_c) \neq 0 \;\text{ for all }\; \Delta_c \in r\mathbf{\Delta}_c\}. \tag{4.49}$$

What happens here? Via the scaling factor r, we inflate or shrink the set $r\mathbf{\Delta}_c$. For small r, $I - M_c\Delta_c$ will be non-singular for any $\Delta_c \in r\mathbf{\Delta}_c$. If r grows larger, we might find some $\Delta_c \in r\mathbf{\Delta}_c$ for which $I - M_c\Delta_c$ will turn out singular. If no such r exists, we have $r_* = \infty$. Otherwise, r_* is just the finite critical value for which we can assure non-singularity for the set $r\mathbf{\Delta}_c$ if r is smaller than r_*. This is the reason why r_* is called *non-singularity margin.*

Remark. r_* also equals the smallest r such that there exists a $\Delta_c \in r\mathbf{\Delta}_c$ that renders $I - M_c\Delta_c$ singular. The above given definition seems more intuitive since we are interested in non-singularity.

Definition 4.11 *The structured singular value (SSV) of the matrix M_c with respect to the set $\mathbf{\Delta}_c$ is defined as*

$$\mu_{\mathbf{\Delta}_c}(M_c) = \frac{1}{r_*} = \frac{1}{\sup\{r \mid \det(I - M_c\Delta_c) \neq 0 \;\textit{ for all }\; \Delta_c \in r\mathbf{\Delta}_c\}}.$$

Remark. The non-singularity margin r_* has been introduced by Michael Safonov, whereas the structured singular value has been defined by John Doyle. Both concepts are equivalent; the structured singular can be related in a nicer fashion to the ordinary singular value what motivates its definition as the reciprocal of r_*.

Let us now assume that we can compute the SSV. Then we can decide the original question whether (4.48) is true or not as follows: We just have to check whether $\mu_{\boldsymbol{\Delta_c}}(M_c) \leq 1$. If yes, then (4.48) is true, if no, then (4.48) is not true. This is the most important fact to remember about the SSV.

Theorem 4.12 *Let M_c be a complex matrix and $\boldsymbol{\Delta}_c$ an arbitrary (open) set of complex matrices. Then*

- $\mu_{\boldsymbol{\Delta_c}}(M_c) \leq 1$ *implies that $I - M_c\Delta_c$ is non-singular for all $\Delta_c \in \boldsymbol{\Delta_c}$.*
- $\mu_{\boldsymbol{\Delta_c}}(M_c) > 1$ *implies that there exists a $\Delta_c \in \boldsymbol{\Delta_c}$ for which $I - M_c\Delta_c$ is singular.*

Proof. Let us first assume that $\mu_{\boldsymbol{\Delta_c}}(M_c) \leq 1$. This implies that $r_* \geq 1$. Suppose that there exists a $\Delta_0 \in \boldsymbol{\Delta_c}$ that renders $I - M_c\Delta_0$ singular. Since $\boldsymbol{\Delta}_c$ is relative open, Δ_0 also belongs to $r\boldsymbol{\Delta}_c$ for some $r < 1$ that is close to 1. By the definition of r_*, this implies that r_* must be smaller than r. Therefore, we conclude that $r_* < 1$ what is a contradiction.

Suppose now that $\mu_{\boldsymbol{\Delta_c}}(M_c) > 1$. This implies $r_* < 1$. Suppose $I - M_c\Delta_c$ is non-singular for all $\Delta_c \in r\boldsymbol{\Delta}_c$ for $r = 1$. This would imply (since r_* was the largest among all r for which this property holds) that $r_* \geq r = 1$, a contradiction.

■

It is important to note that the number $\mu_{\boldsymbol{\Delta_c}}(M_c)$ is depending both on the matrix M_c and on the set $\boldsymbol{\Delta}_c$ what we explicitly indicate in our notation. For the computation of the SSV, the μ-tools expect, as well, a complex matrix `M` and the block structure `blk` as an input. In principle, one might wish to calculate the SSV exactly. Unfortunately, it has been shown through examples that this is a very hard problem in a well-defined sense introduced in computer science. Fortunately, one can calculate a *lower bound* and an *upper bound* for the SSV pretty efficiently. In the μ-tools, this computation is performed with the command `mu(M,blk)` which returns the row `[upperbound lowerbound]`.

For the reader's convenience we explicitly formulate the detailed conclusions that can be drawn if having computed a lower and an upper bound of the SSV.

Theorem 4.13 *Let M_c be a complex matrix and $\boldsymbol{\Delta}_c$ an arbitrary (open) set of complex matrices. Then*

- $\mu_{\boldsymbol{\Delta_c}}(M_c) \leq \gamma_1$ *implies that $I - M_c\Delta_c$ is nonsingular for all $\Delta_c \in \frac{1}{\gamma_1}\boldsymbol{\Delta_c}$.*

- $\mu_{\mathbf{\Delta_c}}(M_c) > \gamma_2$ *implies that there exists a* $\Delta_c \in \frac{1}{\gamma_2}\mathbf{\Delta_c}$ *for which* $I - M_c\Delta_c$ *is singular.*

This is a straightforward consequence of the following simple fact:

$$\alpha\, \mu_{\mathbf{\Delta_c}}(M_c) = \mu_{\mathbf{\Delta_c}}(\alpha M_c) = \mu_{\alpha\mathbf{\Delta_c}}(M_c). \tag{4.50}$$

A scalar scaling of the SSV with factor α is equivalent to scaling either the matrix M_c or the set $\mathbf{\Delta}_c$ with the same factor.

Let us briefly look as well at the case of *matrix valued scalings*. Suppose U and V are arbitrary complex matrices. Then we observe

$$\det(I - M_c[U\Delta_c V]) \neq 0 \ \text{ if and only if } \ \det(I - [VM_cU]\Delta_c) \neq 0. \tag{4.51}$$

This shows that

$$\mu_{U\mathbf{\Delta_c}V}(M_c) = \mu_{\mathbf{\Delta_c}}(VM_cU). \tag{4.52}$$

Hence, if we intend to calculate the SSV with respect to the set

$$U\mathbf{\Delta}_c V = \{U\Delta_c V \mid \Delta_c \in \mathbf{\Delta}_c\},$$

we can do that by calculating the SSV of VM_cU with respect to the original set $\mathbf{\Delta}_c$, and this latter task can be accomplished with the μ-tools.

Before we proceed to a more extended discussion of the background on the SSV, let us discuss its most important purpose, the application to robust stability analysis.

4.9.2 SSV Applied to Testing Robust Stability

For robust stability we had to check (4.47). If we recall Theorem 4.13, this condition holds true if and only if the SSV of $M(i\omega)$ calculated with respect to $\mathbf{\Delta}_c$ is smaller than 1. Since this has to be true for all frequencies, we immediately arrive at the following fundamental result of these lecture notes.

Theorem 4.14 *$I - M\Delta$ has a proper and stable inverse for all $\Delta \in \mathbf{\Delta}$ if and only if*

$$\mu_{\mathbf{\Delta_c}}(M(i\omega)) \leq 1 \ \ \textit{for all} \ \ \omega \in \mathbb{R} \cup \{\infty\}. \tag{4.53}$$

We can again combine Theorem 4.8 with Theorem 4.4 to obtain the test for the general interconnection.

Corollary 4.15 *If K stabilizes P, and if*

$$\mu_{\mathbf{\Delta_c}}(M(i\omega)) \leq 1 \ \ \textit{for all} \ \ \omega \in \mathbb{R} \cup \{\infty\},$$

then K robustly stabilizes $S(\Delta, P)$ against $\mathbf{\Delta}$.

Remark. In case that
$$\mathbf{\Delta}_c := \{\Delta \in \mathbb{C}^{p\times q} \mid \|\Delta\| < 1\}$$
consists of full block matrices only (what corresponds to $n_r = 0$, $n_c = 0$, $n_f = 1$), it follows from the discussion in Section 4.8.3 that
$$\mu_{\mathbf{\Delta_c}}(M(i\omega)) = \|M(i\omega)\|.$$
Hence Theorem 4.14 and Corollary 4.15 specialize to Theorem 4.8 and Corollary 4.9 in this particular case of full block uncertainties.

How do we apply the tests? We just calculate the number $\mu_{\mathbf{\Delta_c}}(M(i\omega))$ for each frequency and check whether it does does not exceed one. In practice, one simply plots the function
$$\omega \to \mu_{\mathbf{\Delta_c}}(M(i\omega))$$
by calculating the right-hand side for finitely many frequencies ω. This allows to visually check whether the curve stays below 1. If the answer is yes, we can conclude robust stability as stated in Theorem 4.14 and Corollary 4.15 respectively. If the answer is no, we reveal in the next section how to determine a destabilizing uncertainty.

In this ideal situation we assume that the SSV can be calculated exactly. As mentioned above, however, only upper bounds can be computed efficiently. Still, with the upper bound it is not difficult to guarantee robust stability. In fact, with a plot of the computed upper bound of $\mu_{\mathbf{\Delta_c}} M(i\omega)$ over the frequency ω, we can easily determine a number $\gamma > 0$ such that
$$\mu_{\mathbf{\Delta_c}}(M(i\omega)) \leq \gamma \;\text{ for all }\; \omega \in \mathbb{R} \cup \{\infty\} \tag{4.54}$$
is satisfied. As before, we can conclude robust stability for the uncertainty set
$$\{\frac{1}{\gamma}\Delta \mid \Delta \in \mathbf{\Delta}\} \tag{4.55}$$
which consists of all uncertainties that admit the same structure as those in $\mathbf{\Delta}$, but that are rather bounded by $\frac{1}{\gamma}$ instead of 1. This is an immediate consequence of Theorem 4.13.

We observe that the SSV-plot or a plot of the upper bound lets us decide the question of *how large we can let the structured uncertainties grow in order to still infer robust stability*.

If varying γ, the largest class $\frac{1}{\gamma}\mathbf{\Delta}$ is obtained with the smallest γ for which (4.54) is valid; this best value is clearly given as
$$\gamma_* = \sup_{\omega \in \mathbb{R}\cup\{\infty\}} \mu_{\mathbf{\Delta_c}}(M(i\omega)).$$
Since $\frac{1}{\gamma_*}$ is the largest possible inflating factor for the set of uncertainties, this number is often called *stability margin*.

4.9.3 Construction of Destabilizing Perturbations

Let us suppose that we have found some frequency $\omega_0 \in (0, \infty)$ for which

$$\mu_{\mathbf{\Delta_c}}(M(i\omega_0)) > \gamma_0.$$

Such a pair of frequency ω_0 and value γ_0 can be found by visually inspecting a plot of the lower bound of the SSV over frequency as delivered by the μ-tools. Due to Theorem 4.13, there exists some $\Delta_0 \in \frac{1}{\gamma_0}\mathbf{\Delta}_c$ that renders $I - M(i\omega_0)\Delta_0$ singular. Note that the algorithm in the μ-tools to compute a lower bound of $\mu_{\mathbf{\Delta_c}}(M(i\omega_0))$ returns such a matrix Δ_0 for the calculated bound γ_0.

Based on ω_0, γ_0 and Δ_0, we intend to point out in this section how we can determine a candidate for a dynamic destabilizing perturbation as discussed in Section 4.7.4.

Let us denote the blocks of Δ_0 as

$$\Delta_0 = \begin{pmatrix} p_1 I & & & & & & & & & 0 \\ & \ddots & & & & & & & & \\ & & p_{n_r} I & & & & & & & \\ & & & \delta_1^0 I & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \delta_{n_c}^0 I & & & & \\ & & & & & & \Delta_1^0 & & & \\ & & & & & & & \ddots & & \\ 0 & & & & & & & & & \Delta_{n_f}^0 \end{pmatrix} \in \mathbb{C}^{p\times q}$$

with $p_j \in \mathbb{R}$, $\delta_j^0 \in \mathbb{C}$, $\Delta_j^0 \in \mathbb{C}^{p_j \times q_j}$.

According to Lemma **??**, there exists proper and stable $\delta_j(s)$ with

$$\delta_j(i\omega_0) = \delta_j^0 \quad \text{and} \quad \|\delta_j\|_\infty \le |\delta_j^0| < \frac{1}{\gamma_0}.$$

Since $I - M(i\omega_0)\Delta_0$ is singular, there exists a complex kernel vector $u \neq 0$ with $(I - M(i\omega_0)\Delta_0)u = 0$. Define $v = \Delta_0 u$. If we partition u and v according to Δ_0, we obtain $v_j = \Delta_j^0 u_j$ for those vector pieces that correspond to the full blocks. In the proof of Theorem 4.10 we have shown how to construct a real rational proper and stable $\Delta_j(s)$ that satisfies

$$\Delta_j(i\omega_0) = v_j \frac{u_j^*}{\|u_j\|^2} \quad \text{and} \quad \|\Delta_j\|_\infty \le \frac{\|v_j\|}{\|u_j\|} \le \|\Delta_j^0\| < \frac{1}{\gamma_0}.$$

(Provide additional arguments if it happens that $u_j = 0$).

Let us then define the proper and stable dynamic perturbation

$$\Delta(s) = \begin{pmatrix} p_1 I & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & p_{n_r} I & & & & & & \\ & & & \delta_1(s) I & & & & & \\ & & & & \ddots & & & & \\ & & & & & \delta_{n_c}(s) I & & & \\ & & & & & & \Delta_1(s) & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & \Delta_{n_f}(s) \end{pmatrix}.$$

Since each of the diagonal blocks has an H_∞-norm that does not exceed γ_0, we infer $\|\Delta\|_\infty < \frac{1}{\gamma_0}$. Hence $\Delta \in \frac{1}{\gamma_0}\mathbf{\Delta}$. Moreover, by inspection one verifies that $\Delta(i\omega_0)u = v$. This implies that $[I - M(i\omega_0)\Delta(i\omega_0)]u = u - M(i\omega_0)v = 0$ such that $I - M(s)\Delta(s)$ has a zero at $i\omega_0$ and, hence, its inverse is certainly not stable.

If $\omega_0 = 0$ or $\omega_0 = \infty$, this construction fails since δ_j^0 and Δ_j^0 cannot, in general, taken to be real. We refer to the reference

A.L. Tits, M.K.H. Fan, On the small-μ theorem, Automatica **31** (1995) 1199-1201

for a construction of Δ under these circumstances.

In summary, we have found an uncertainty Δ that is destabilizing for $(I - M\Delta)^{-1}$, and that is a candidate for rendering the system $S(\Delta, P)$ controlled with K unstable.

4.9.4 Example: Two Uncertainty Blocks in Tracking Configuration

Let us come back to Figure 41 with

$$G(s) = \frac{200}{(10s+1)(0.05s+1)^2} \quad \text{and} \quad K(s) = 0.2\frac{0.1s+1}{(0.65s+1)(0.03s+1)}.$$

In this little example we did not include any weightings for the uncertainties what is, of course, unrealistic. Note that the uncertainties Δ_1 and Δ_2 have both dimension 1×1. Pulling them out leads to the uncertainty

$$\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}$$

for the interconnection. Since both uncertainties are dynamic, we infer for this setup that we have to choose

$$\mathbf{\Delta}_c := \{ \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \mid \Delta_j \in \mathbb{C},\ |\Delta_j| < 1,\ j = 1, 2\}.$$

For any complex matrix `M`, the command `mu(M, [1 1; 1 1])` calculates the SSV of M with respect to $\mathbf{\Delta}_c$. The matrix `[1 1; 1 1]` just indicates that the structure consists of two blocks (two rows) that are both of dimension 1×1.

The code

```
G=nd2sys( [200],conv([10 1],conv([0.05 1],[0.05 1])) );
K=nd2sys( [0.1 1], conv([0.65 1],[0.03 1]),0.2 );

systemnames='G';
inputvar='[w1;w2;d;n;r;u]';
outputvar='[u;r-n-d-w1-G;G+w1+d-r;r+w2-n-d-w1-G]';
input_to_G='[u]';
sysoutname='P';
cleanupsysic='yes';
sysic
N=starp(P,K);
M11=sel(N,1,1);
M22=sel(N,2,2);
M=sel(N,[1 2],[1 2]);
om=logspace(0,1);
clf;
vplot('liv,m',frsp(M11,om),':',frsp(M22,om),':',vnorm(frsp(M,om)),'--');
hold on;grid on
Mmu=mu(frsp(M,om),[1 1;1 1]);
vplot('liv,m',Mmu,'-');
```

computes the transfer matrix M seen by the uncertainty. Note that M_{jj} is the transfer function seen by Δ_j for $j = 1, 2$. We plot $|M_{11}(i\omega)|$, $|M_{22}(i\omega)|$, $\|M(i\omega)\|$, and $\mu_{\mathbf{\Delta_c}}(M(i\omega))$ over the frequency interval $\omega \in [0, 10]$, as shown in Figure 45. For good reasons to be revealed in Section 4.10, the upper bound of the SSV coincides with the lower bound such that, in this example, we have exactly calculated the SSV.

How do we have to interpret this plot? Since the SSV is not larger than 2, we conclude

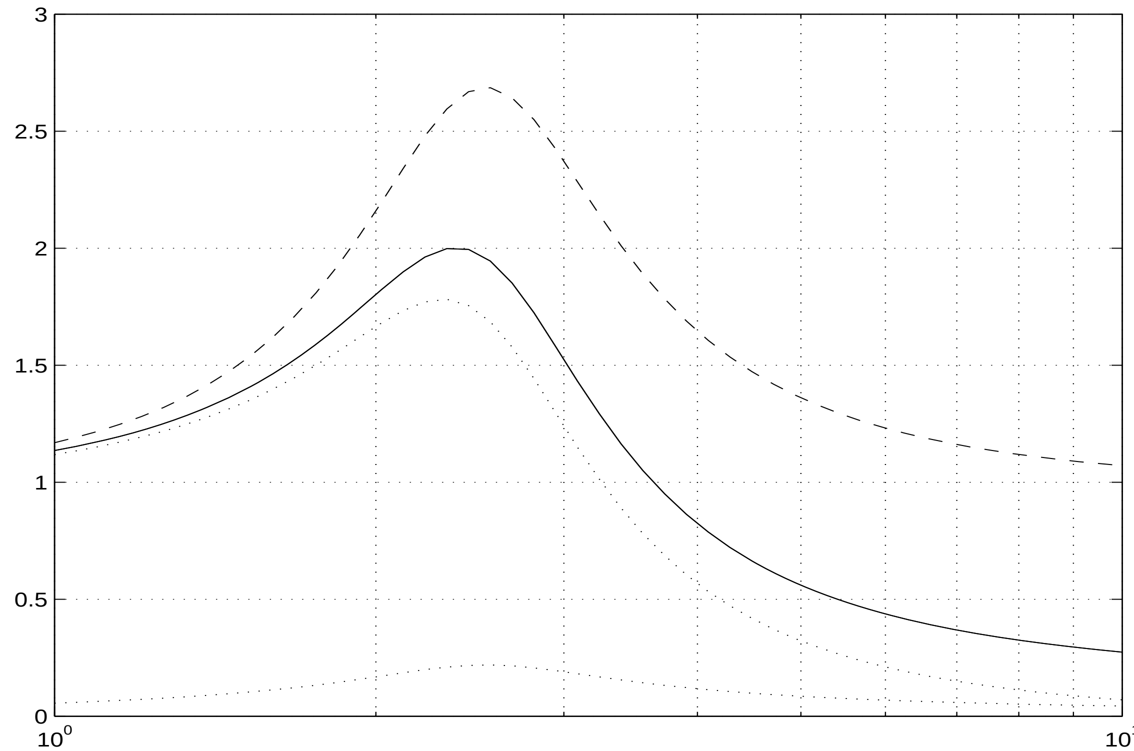


Figure 45: Magnitudes of M_{11}, M_{22} (dotted), norm of M (dashed), SSV of M (solid)

robust stability for all uncertainties that take their values in

$$\frac{1}{2}\mathbf{\Delta}_c = \{\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \mid \Delta_j \in \mathbb{C},\ |\Delta_j| < \frac{1}{2},\ j = 1, 2\}.$$

For this statement, we did not take into account that the SSV plot shows a variation in frequency. To be specific, at the particular frequency ω, we can not only allow for $\|\Delta_c\| < 0.5$ but even for

$$\|\Delta_c\| < r(\omega) := \frac{1}{\mu_{\mathbf{\Delta_c}}(M(i\omega))}$$

and we can still conclude that $I - M(i\omega)\Delta_c$ is non-singular. Therefore, one can guarantee robust stability for uncertainties that take their values at $i\omega$ in the set

$$\mathbf{\Delta}_c(\omega) := \{\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \in \mathbb{C}^{2\times 2} \mid |\Delta_1| < r(\omega),\ |\Delta_2| < r(\omega)\}.$$

The SSV plot only leads to insights if re-scaling the *whole matrix* Δ_c. How can we explore robust stability for *different bounds on the different uncertainty blocks*? This would correspond to uncertainties that take their values in

$$\{\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \in \mathbb{C}^{2\times 2} \mid |\Delta_1| < r_1,\ |\Delta_2| < r_2\}$$

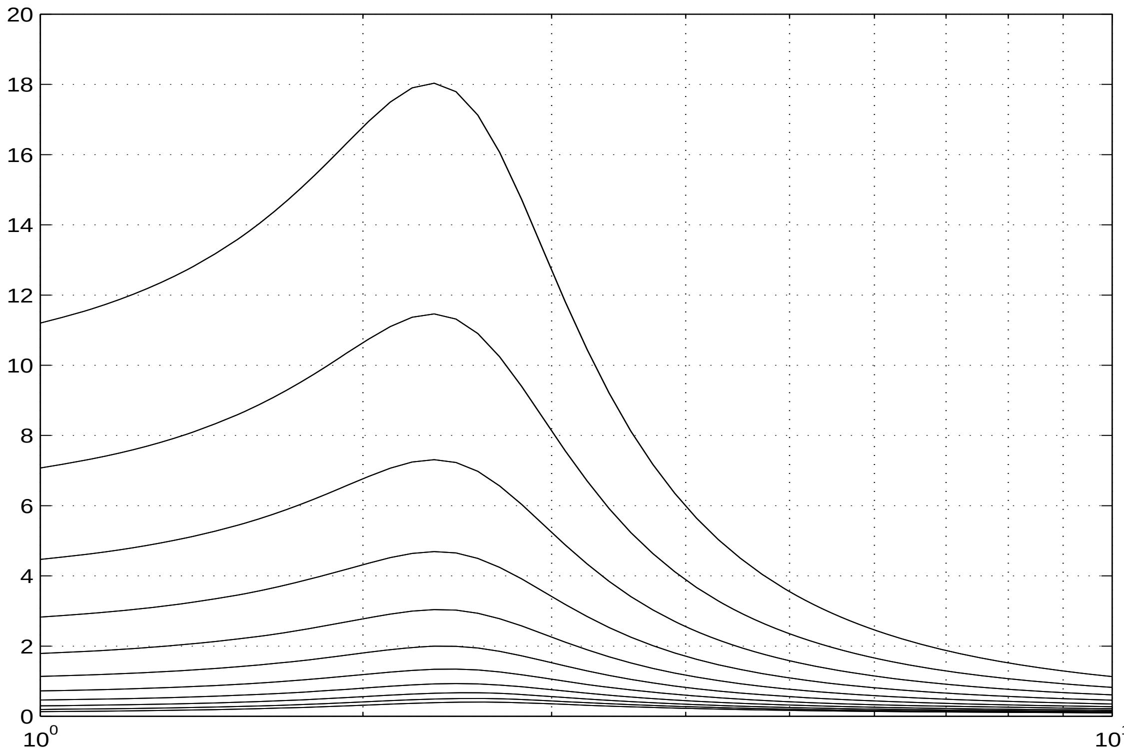


Figure 46: SSV of RM with $r_1 = 1$, $r_2 \in [0.1, 10]$.

for different $r_1 > 0$, $r_2 > 0$. The answer is simple: just employ weightings! Observe that this set is nothing but

$$R\mathbf{\Delta}_c \quad \text{with} \quad R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

In order to guarantee robust stability, we have to test $\mu_{R\mathbf{\Delta_c}}(M(i\omega)) \leq 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$, what amounts to verifying

$$\mu_{\mathbf{\Delta_c}}(RM(i\omega)) < 1 \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}$$

by the property (4.52). Again, we look at our example where we vary r_2 in the interval $[0.1, 10]$ and fix $r_1 = 1$. Figure 46 presents the SSV plots for these values.

Important task. Provide an interpretation of the plots!

Remark. For the last example, we could have directly re-scaled the two uncertainty blocks in the interconnection, and then pulled out the normalized uncertainties. The resulting test will lead, of course, to the same conclusions.

Similar statements can be made for the dashed curve and full block uncertainties; the discussion is related to the set

$$\{\Delta_c = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2} \mid \|\Delta_c\| < 1\}.$$

The dotted curves lead to robust stability results for

$$\{\begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} \mid \Delta_1 \in \mathbb{C},\ |\Delta_1| < 1\}$$

or

$$\{\begin{pmatrix} 0 & 0 \\ 0 & \Delta_2 \end{pmatrix} \mid \Delta_2 \in \mathbb{C},\ |\Delta_2| < 1\}$$

respectively.

Important task. Formulate the exact results and interpretations for the last three cases.

4.9.5 SSV Applied to Testing the General Hypothesis

Let us recall that we always have to verify the hypotheses 4.3 before we apply our results. So far we were not able to check (4.23). For the specific set considered in this section, this simply amounts to a SSV-test: (4.23) is true if and only if

$$\mu_{\mathbf{\Delta_c}}(P_{11}(\infty)) \leq 1.$$

Remark. We observe that the SSV is a tool that is by no means restricted to stability tests in control. In fact, it is useful in any problem where one needs to check whether a family of matrices is non-singular.

4.10 A Brief Survey on the Structured Singular Value

This section serves to provide some important properties of $\mu_{\mathbf{\Delta_c}}(M_c)$, and it should clarify how it is possible to compute bounds on this quantity.

An important property of the SSV is its 'monotonicity' in the set $\mathbf{\Delta}_c$: If two sets of complex matrices $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ satisfy

$$\mathbf{\Delta}_1 \subset \mathbf{\Delta}_2,$$

then we infer

$$\mu_{\mathbf{\Delta_1}}(M_c) \leq \mu_{\mathbf{\Delta_2}}(M_c).$$

In short: The larger the set $\mathbf{\Delta}_c$ (with respect to inclusion), the larger $\mu_{\mathbf{\Delta_c}}(M_c)$.

Now it is simple to understand the basic idea of how to bound the SSV. For that purpose let us introduce the specific sets

$$\begin{aligned} \mathbf{\Delta}_1 &:= \{pI \in \mathbb{R}^{q\times q} \mid |p| < 1\} \\ \mathbf{\Delta}_2 &:= \{\delta I \in \mathbb{C}^{q\times q} \mid |\delta| < 1\} \\ \mathbf{\Delta}_3 &:= \{\Delta_c \in \mathbb{C}^{p\times q} \mid \|\Delta_c\| < 1\} \end{aligned}$$

that correspond to one real repeated block ($n_r = 1$, $n_c = 0$, $n_f = 0$), one complex repeated block ($n_r = 0$, $n_c = 1$, $n_f = 0$), or one full block ($n_r = 0$, $n_c = 0$, $n_f = 1$). For these specific structures one can easily compute the SSV explicitly:

$$\begin{aligned} \mu_{\mathbf{\Delta_1}}(M_c) &= \rho_{\mathbb{R}}(M_c) \\ \mu_{\mathbf{\Delta_2}}(M_c) &= \rho(M_c) \\ \mu_{\mathbf{\Delta_3}}(M_c) &= \|M_c\|. \end{aligned}$$

Here, $\rho_{\mathbb{R}}(M)$ denotes the *real* spectral radius of M_c defined as

$$\rho_{\mathbb{R}}(M_c) = \max\{|\lambda| \mid \lambda \text{ is a real eigenvalue of } M_c\},$$

whereas $\rho(M)$ denotes the *complex* spectral radius of M_c that is given as

$$\rho(M_c) = \max\{|\lambda| \mid \lambda \text{ is a complex eigenvalue of } M_c\}.$$

In general, we clearly have

$$\mathbf{\Delta}_1 \subset \mathbf{\Delta}_c \subset \mathbf{\Delta}_3$$

such that we immediately conclude

$$\mu_{\mathbf{\Delta_1}}(M_c) \le \mu_{\mathbf{\Delta_c}}(M_c) \le \mu_{\mathbf{\Delta_3}}(M_c).$$

If there are *no real blocks* ($n_r = 0$), we infer

$$\mathbf{\Delta}_2 \subset \mathbf{\Delta}_c \subset \mathbf{\Delta}_3$$

what implies

$$\mu_{\mathbf{\Delta_2}}(M_c) \le \mu_{\mathbf{\Delta_c}}(M_c) \le \mu_{\mathbf{\Delta_3}}(M_c).$$

Together with the above given explicit formulas, we arrive at the following result.

Lemma 4.16 *In general,*

$$\rho_{\mathbb{R}}(M_c) \le \mu_{\mathbf{\Delta_c}}(M_c) \le \|M_c\|.$$

If $n_r = 0$, then

$$\rho(M_c) \le \mu_{\mathbf{\Delta_c}}(M_c) \le \|M_c\|.$$

Note that these bounds are pretty rough. The main goal in computational techniques is to refine these bounds to get close to the actual value of the SSV.

4.10.1 Continuity

We have seen that the SSV of M_c with respect to the set $\{pI \in \mathbb{R}^{p\times q} \mid |p| < 1\}$ is the real spectral radius. This reveals that the SSV does, in general, not depend continuously on M_c. Just look at the simple example

$$\rho_{\mathbb{R}} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} = \begin{cases} 0 & \text{for} \quad m \neq 1 \\ 1 & \text{for} \quad m = 0. \end{cases}$$

It shows that the value of the SSV can jump with only slight variations in m.

This is an important observation for practice. If the structure (4.46) comprises real blocks ($n_r \neq 0$), then

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega))$$

might have *jumps* if we vary ω. Even more dangerously, since we compute the SSV at only a finite number of frequencies, we might miss a frequency where the SSV jumps to very high levels. The plot could make us believe that the SSV is smaller than one and we would conclude robust stability; in reality, the plot jumps above one at some frequency which we have missed, and the conclusion was false.

The situation is more favorable if there are no real blocks $n_r = 0$. Then $\mu_{\boldsymbol{\Delta_c}}(M_c)$ depends continuously on M_c, and jumps do not occur.

Theorem 4.17 *If $n_r = 0$ such that no real blocks appear in (4.46), the function*

$$M_c \to \mu_{\boldsymbol{\Delta_c}}(M_c)$$

is continuous. In particular, If M is real-rational, proper and stable,

$$\omega \to \mu_{\boldsymbol{\Delta_c}}(M(i\omega))$$

defines a continuous function on $\mathbb{R} \cup \{\infty\}$.

4.10.2 Lower Bounds

If one can compute some

$$\Delta_0 \in \frac{1}{\gamma}\boldsymbol{\Delta}_c \ \text{ that renders } \ I - M_c\Delta_0 \ \text{ singular,} \tag{4.56}$$

one can conclude that

$$\gamma \leq \mu_{\boldsymbol{\Delta_c}}(M_c).$$

The approach taken in the μ-tools is to maximize γ such that there exists a Δ_0 as in (4.56). There is no guarantee whether one can compute the global optimum for the resulting maximization problem. Nevertheless, any step in increasing the value γ improves the lower bound and is, hence, beneficial.

Note that the algorithm outputs a matrix Δ_0 as in (4.56) for the best achievable lower bound γ. Based on this matrix Δ_0, one can compute a destabilizing perturbation as described in Section 4.9.3.

If the structure (4.46) only comprises real blocks ($n_c = 0$, $n_f = 0$), it often happens that the algorithm fails and that the lower bound is actually just zero. In general, if real blocks in the uncertainty structure do exist ($n_r \neq 0$), the algorithm is less reliable if compared to the case when these blocks do not appear ($n_r = 0$). We will not go into the details of these quite sophisticated algorithms.

4.10.3 Upper Bounds

If one can test that

$$\text{for all } \ \Delta_c \in \frac{1}{\gamma}\mathbf{\Delta}_c \ \text{ the matrix } \ I - M_c\Delta_c \ \text{ is non-singular,} \tag{4.57}$$

one can conclude that

$$\mu_{\mathbf{\Delta_c}}(M_c) \leq \gamma.$$

We have already seen in Section (4.8.2) that $\|M_c\| \leq \gamma$ is a sufficient condition for (4.57) to hold.

How is it possible to refine this condition?

Simple Scalings

Let us assume that all the full blocks in (4.46) are square such that $p_j = q_j$. Suppose that D is any non-singular matrix that satisfies

$$D\Delta_c = \Delta_c D \ \text{ for all } \ \Delta_c \in \frac{1}{\gamma}\mathbf{\Delta}_c. \tag{4.58}$$

Then

$$\|D^{-1}M_cD\| < \gamma \tag{4.59}$$

implies that

$$I - [D^{-1}M_cD]\Delta_c \tag{4.60}$$

is non-singular for all $\Delta_c \in \frac{1}{\gamma}\mathbf{\Delta}_c$. If we exploit $D\Delta_c = \Delta_c D$, (4.60) is nothing but

$$I - D^{-1}[M_c\Delta_c]D = D^{-1}[I - M_c\Delta_c]D.$$

Therefore, not only (4.60) but even $I - M_c\Delta_c$ itself is non-singular. This implies that (4.57) is true such that γ is an upper bound for $\mu_{\mathbf{\Delta_c}}(M_c)$.

In order to find the *smallest* upper bound, we hence need to minimize

$$\|D^{-1}M_cD\|$$

over the set of all matrices D that satisfy (4.58). Since $D = I$ is in the class of all these matrices, the minimal value is certainly better than $\|M_c\|$, and we can indeed possibly refine this rough upper bound through the introduction of the extra variables D. Since the object of interest is a scaled version $D^{-1}M_cD$ of M_c, these variables D are called *scalings*. Let us summarize what we have found so far.

Lemma 4.18 *We have*

$$\mu_{\mathbf{\Delta_c}}(M_c) \leq \inf_{D \text{ satisfies } (4.58) \text{ and is non-singular}} \|D^{-1}M_cD\|.$$

In order to find the best upper bound, we have to solve the minimization problem on the right. Both from a theoretical and a practical view-point, it has very favorable properties: It is a convex optimization problem for which fast solvers are available. Convexity implies that one can really find the global optimum.

In these notes we only intend to reveal that the fundamental reason for the favorable properties can be attributed to the following fact: Finding a non-singular D with (4.58) and (4.59) is a so-called **L**inear **M**atrix **I**nequality (LMI) problem. For such problems very efficient algorithms have been developed in recent years.

As a first step it is very simple to see that (4.58) holds if and only if D admits the structure

$$D = \begin{pmatrix} D_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & D_{n_r} & & & & & & \\ & & & D_{n_r+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & D_{n_r+n_c} & & & \\ & & & & & & d_1 I & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & d_{n_f} I \end{pmatrix} \tag{4.61}$$

with

$$D_j \;\text{ a non-singular complex matrix and }\; d_j \;\text{ a non-zero complex scalar}$$

of the same size as the corresponding blocks in the partition (4.46). It is interesting to observe that any repeated block in (4.46) corresponds to a full block in (4.61), and any full block in (4.46) corresponds to a repeated block in (4.61).

In a second step, one transforms (4.59) into a linear matrix inequality: (4.59) is equivalent to

$$[D^{-1}M_cD][D^{-1}M_cD]^* < \gamma^2 I.$$

If we left-multiply with D and right-multiply with D^*, we arrive at the equivalent inequality

$$M_c[DD^*]M_c^* < \gamma^2[DD^*].$$

Let us introduce the Hermitian matrix

$$Q := DD^*$$

such that the inequality reads as

$$M_cQM_c^* < \gamma^2 Q. \tag{4.62}$$

Moreover, Q has the structure

$$Q = \begin{pmatrix} Q_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & Q_{n_r} & & & & & & \\ & & & Q_{n_r+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & Q_{n_r+n_c} & & & \\ & & & & & & q_1 I & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & q_{n_f} I \end{pmatrix} \tag{4.63}$$

with

$$Q_j \quad \text{a Hermitian positive definite matrix and} \quad q_j \quad \text{a real positive scalar.} \tag{4.64}$$

Testing whether there exists a Q with the structure (4.63) that satisfies the matrix inequality (4.62) is an LMI problem.

Here we have held γ fixed. Typical LMI algorithms also allow to directly minimize γ in order to find the best upper bound. Alternatively, one can resort to the standard bisection algorithm as discussed in Section A.

A Larger Class of Scalings

Clearly, the larger the class of considered scalings, the more freedom is available to approach the actual value of the SSV. Hence a larger class of scalings might lead to the computation of better upper bounds.

These arguments turn out to be valid, in particular, if the structure (4.46) comprises real blocks. The fundamental idea to arrive at better upper bounds is formulated in the following simple lemma.

Lemma 4.19 *Suppose there exists a Hermitian P such that*

$$\begin{pmatrix} \Delta_c \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta_c \\ I \end{pmatrix} \geq 0 \ \text{ for all } \ \Delta_c \in \frac{1}{\gamma}\mathbf{\Delta}_c \tag{4.65}$$

and that satisfies

$$\begin{pmatrix} I \\ M_c \end{pmatrix}^* P \begin{pmatrix} I \\ M_c \end{pmatrix} < 0. \tag{4.66}$$

Then (4.57) holds true and, hence, γ is an upper bound for $\mu_{\mathbf{\Delta_c}}(M_c)$.

Proof. The proof is extremely simple. Fix $\Delta_c \in \frac{1}{\gamma}\mathbf{\Delta}_c$. We have to show that $I - M_c\Delta_c$ is non-singular. Let us assume the contrary: $I - M_c\Delta_c$ is singular. Then there exists an $x \neq 0$ with $(I - M_c\Delta_c)x = 0$. Define $y := \Delta_c x$ such that $x = M_c y$. Then (4.65) leads to

$$0 \leq x^* \begin{pmatrix} \Delta_c \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta_c \\ I \end{pmatrix} x = \begin{pmatrix} y \\ x \end{pmatrix}^* P \begin{pmatrix} y \\ x \end{pmatrix}.$$

On the other hand, (4.66) implies

$$0 > y^* \begin{pmatrix} I \\ M_c \end{pmatrix}^* P \begin{pmatrix} I \\ M_c \end{pmatrix} y = \begin{pmatrix} y \\ x \end{pmatrix}^* P \begin{pmatrix} y \\ x \end{pmatrix}.$$

(Since $x \neq 0$, the vector $\begin{pmatrix} y \\ x \end{pmatrix}$ is also non-zero.) This contradiction shows that $I - M_c\Delta_c$ cannot be singular.

■

Remark. In Lemma 4.19 the converse holds as well: If the SSV is smaller than γ, there exists a Hermitian P with (4.65) and (4.66). In principle, based on this lemma, one could compute the exact value of the SSV; the only crux is to parametrize the set of all scalings P with (4.65) what cannot be achieved in an efficient manner.

Remark. In order to work with (4.58) in the previous section, we need to assume that the full blocks Δ_j are square. For (4.65) no such condition is required. The μ-tools can handle non-square blocks as well.

For practical applications, we need to find a set of scalings that all fulfill (4.65). A very straightforward choice that is implemented in the μ-tools is as follows: Let $\mathcal{P}_\gamma$ consist of all matrices

$$P = \begin{pmatrix} -\gamma^2 Q & S \\ S^* & Q \end{pmatrix}$$

where Q has the structure (4.63) and (4.64), and S is given by

$$S = \begin{pmatrix} S_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & S_{n_r} & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 0 \end{pmatrix} \tag{4.67}$$

with

$$S_j \ \text{ a complex skew-Hermitian matrix: } \ S_j^* + S_j = 0.$$

For any $P \in \mathcal{P}_\gamma$, the matrix $\begin{pmatrix} \Delta_c \\ I \end{pmatrix}^* P \begin{pmatrix} \Delta_c \\ I \end{pmatrix}$ is block-diagonal. Let us now check for each diagonal block that it is positive semidefinite: We observe for

- real uncertainty blocks:

$$p_j(-\gamma^2 Q_j)p_j + p_j S_j + S_j^* p_j + Q_j = Q_j(-\gamma^2 p_j^2 + 1) \geq 0.$$

- complex repeated blocks:

$$\delta_j^*(-\gamma^2 Q_j)\delta_j + Q_j = Q_j(-\gamma^2 |\delta_j|^2 + 1) \geq 0.$$

- complex full blocks:

$$\Delta_j^*(-\gamma^2 q_j I)\Delta_j + q_j I = q_j(-\gamma^2 \Delta_j^*\Delta_j + I) \geq 0.$$

We conclude that (4.65) holds for any $P \in \mathcal{P}_\gamma$.

If we can find one $P \in \mathcal{P}_\gamma$ for which also condition (4.66) turns out to be true, we can conclude that γ is an upper bound for the SSV. Again, testing the existence of $P \in \mathcal{P}_\gamma$ that also satisfies (4.66) is a standard LMI problem.

The best upper bound is of course obtained as follows:

$$\text{Minimize } \gamma \text{ such that there exists a } P \in \mathcal{P}_\gamma \text{ that satisfies (4.66).} \tag{4.68}$$

Again, the best bound can be compute by bisection as described in Section A.

Remark. Only if real blocks do exist, the matrix S will be non-zero, and only in that case we will benefit from the extension discussed in this section. We have described the class of scalings that is employed in the μ-tools. However, Lemma 4.19 leaves room for considerable improvements in calculating upper bounds for the SSV.

4.11 When is the Upper Bound equal to the SSV?

Theorem 4.20 *If*

$$2(n_r + n_c) + n_f \leq 3,$$

then the SSV $\mu_{\mathbf{\Delta_c}}(M_c)$ is not only bounded by but actually coincides *with the optimal value of problem (4.68).*

Note that this result is tight in the following sense: If $2(n_r + n_c) + n_f > 3$, one can construct examples for which there is a gap between the SSV and the best upper bound, the optimal value of (4.68).

4.11.1 Example: Different Lower and Upper Bounds

Let

$$M(s) = \begin{pmatrix} \frac{1}{2s+1} & 1 & \frac{s-2}{2s+4} \\ -1 & \frac{s}{s^2+s+1} & \frac{1}{(s+1)^2} \\ \frac{3s}{s+5} & \frac{-1}{4s+1} & 1 \end{pmatrix}.$$

Moreover, consider three different structures.

The first consists of two full complex blocks

$$\mathbf{\Delta}_1 := \left\{ \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \mid \Delta_1 \in \mathbb{C},\ |\Delta_1| < 1,\ \Delta_2 \in \mathbb{C}^{2\times 2},\ \|\Delta_2\| < 1 \right\}.$$

We plot in Figure 47 $\|M(i\omega)\|$, $\mu_{\mathbf{\Delta_1}}(M(i\omega))$, $\rho(M(i\omega))$ over frequency.

We conclude that $I - M(i\omega_0)\Delta_c$ is non-singular for all

- full blocks $\Delta_c \in \mathbb{C}^{3\times 3}$ with $\|\Delta_c\| < \frac{1}{\gamma_1}$

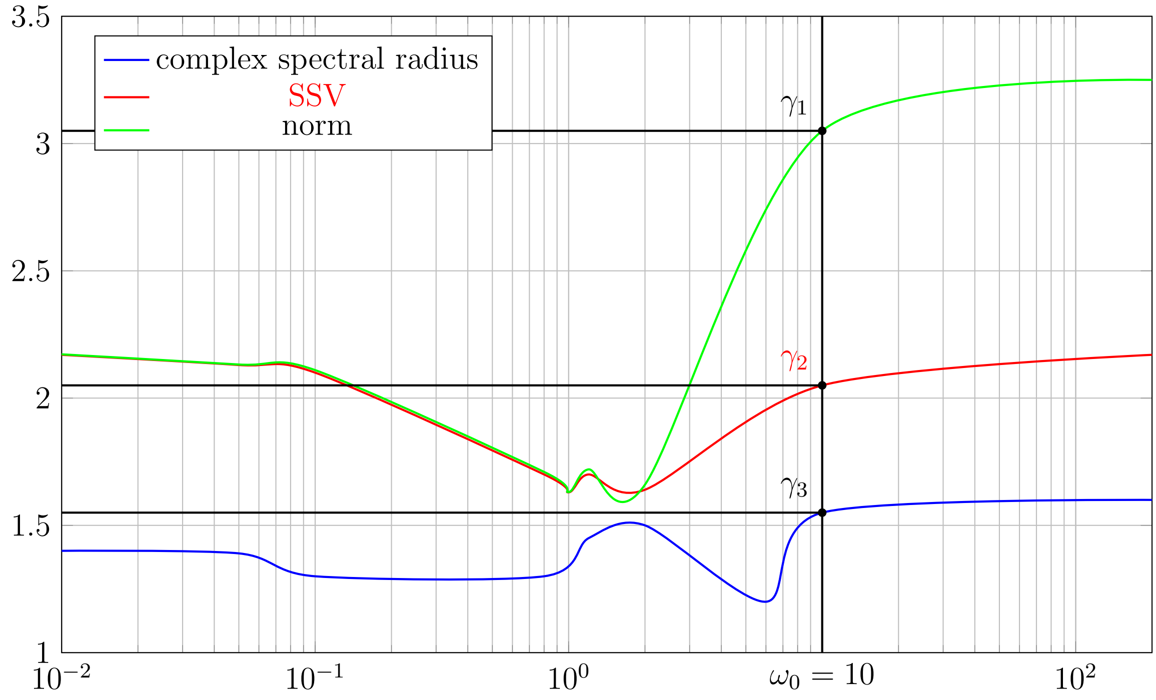


Figure 47: Plots of $\|M(i\omega)\|$, $\mu_{\boldsymbol{\Delta_1}}(M(i\omega))$ and $\rho(M(i\omega))$ over frequency.

- structured blocks $\Delta_c \in \frac{1}{\gamma_2}\boldsymbol{\Delta}_1$
- complex repeated blocks $\begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix}$ with $|\delta| < \frac{1}{\gamma_3}$.

In addition, there exists a Δ_c that is

- a full block $\Delta_c \in \mathbb{C}^{3\times 3}$ with $\|\Delta_c\| < \frac{1}{\gamma}$, $\gamma < \gamma_1$
- a structured block $\Delta_c \in \frac{1}{\gamma}\boldsymbol{\Delta}_1$, $\gamma < \gamma_2$
- a complex repeated block $\begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix}$ with $|\delta| < \frac{1}{\gamma}$, $\gamma < \gamma_3$

that renders $I - M(i\omega_0)\Delta_c$ singular.

As a second case, we take one repeated complex block and one full complex block:

$$\boldsymbol{\Delta}_2 := \left\{ \begin{pmatrix} \Delta_1 & 0 \\ 0 & \delta_2 I_2 \end{pmatrix} \mid \Delta_1 \in \mathbb{C},\ |\Delta_1| < 1,\ \delta_2 \in \mathbb{C},\ |\delta_2| < 1 \right\}.$$

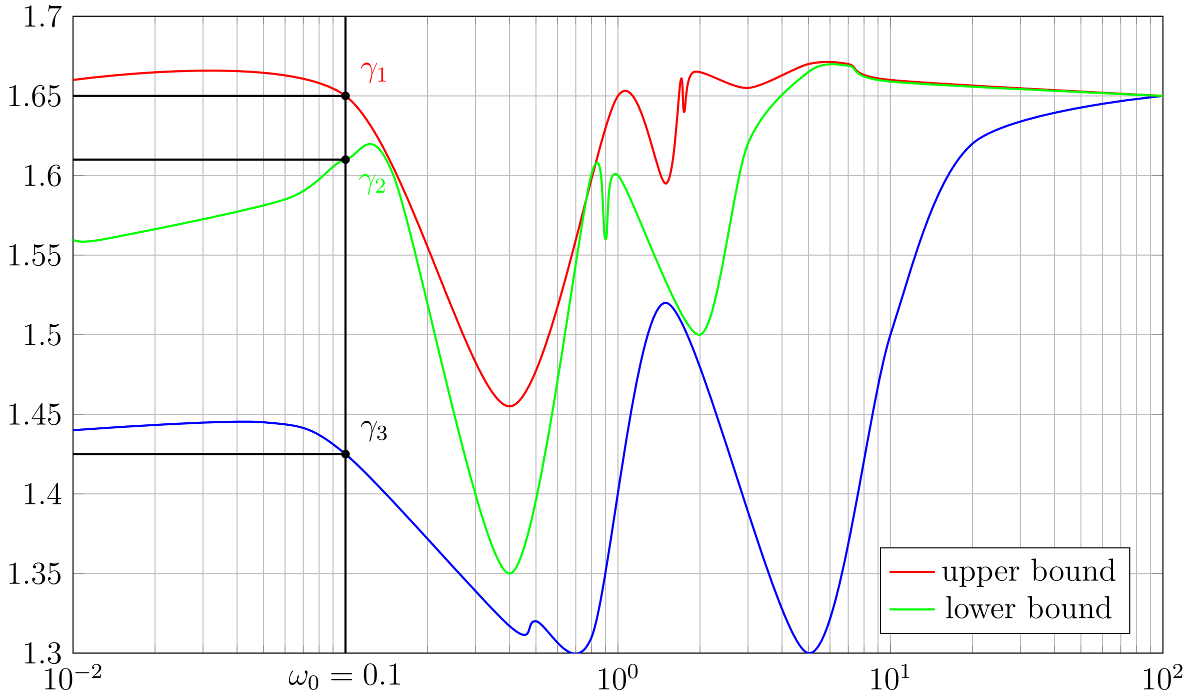


Figure 48: Plots of bounds on $\mu_{\mathbf{\Delta_1}}(M(i\omega))$ and of $\rho(M(i\omega))$ over frequency.

As observed in Figure 48, the upper and lower bounds for the SSV are *different*. Still, the complex spectral radius bounds the lower bound on the SSV from below.

We conclude that $I - M(i\omega_0)\Delta_c$ is non-singular for all structured Δ_c in $\frac{1}{\gamma_1}\mathbf{\Delta}_2$. There exists a structured Δ_c in $\frac{1}{\gamma}\mathbf{\Delta}_2$, $\gamma < \gamma_2$, that renders $I - M(i\omega_0)\Delta_c$ singular.

As a last case, let us consider a structure with one real repeated block and one full complex block:

$$\mathbf{\Delta}_3 := \left\{ \begin{pmatrix} \Delta_1 & 0 \\ 0 & \delta_2 I_2 \end{pmatrix} \mid \Delta_1 \in \mathbb{C},\ |\Delta_1| < 1,\ \delta_2 \in \mathbb{R},\ |\delta_2| < 1 \right\}.$$

Figure 49 shows that lower bound and upper bound of the SSV are further apart than in the previous example, what reduces the quality of the information about the SSV. Since the structure comprises a real block, the complex spectral radius is no lower bound on the SSV (or its lower bound) over all frequencies, as expected.

The upper bound is smaller than γ_1 for all frequencies. Hence $I - M\Delta$ has proper and stable inverse for all $\Delta \in \frac{1}{\gamma_1}\mathbf{\Delta}$.

There exists a frequency for which the lower bound is larger that γ_2: Hence there exists a $\Delta \in \frac{1}{\gamma_2}\mathbf{\Delta}$ such that $I - M\Delta$ does not have a proper and stable inverse.

We emphasize that the

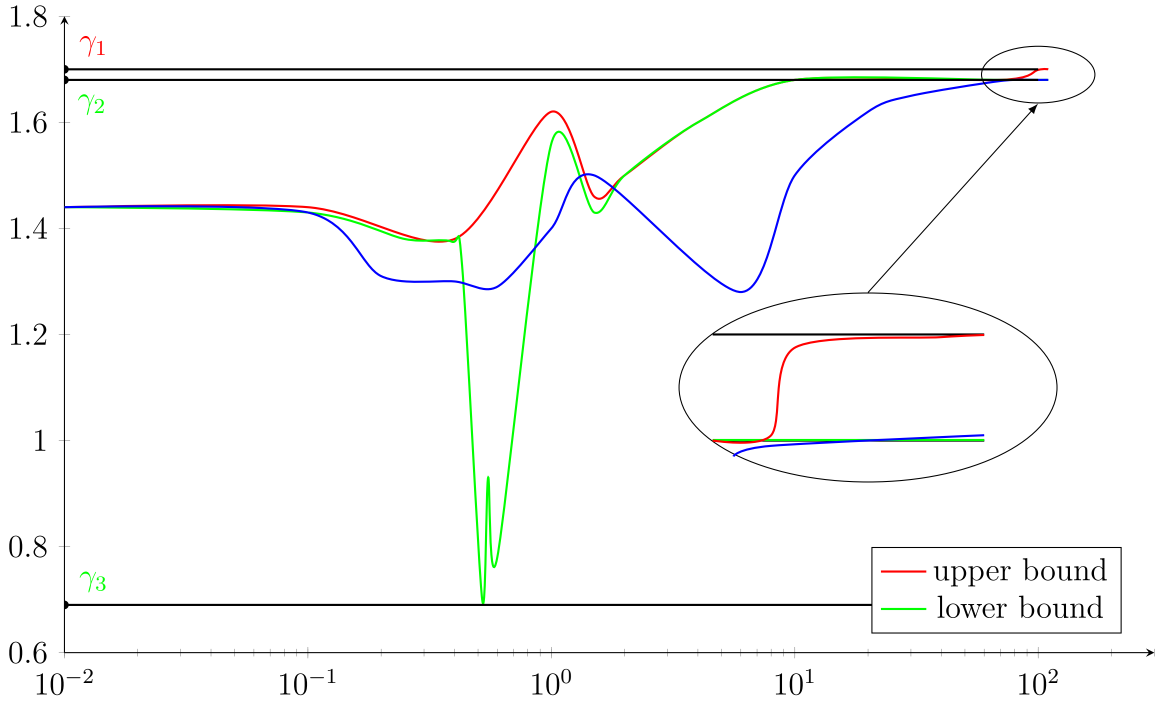


Figure 49: Plots of bounds on $\mu_{\mathbf{\Delta_1}}(M(i\omega))$ and of $\rho(M(i\omega))$ over frequency.

Exercises

1) This is a continuation of Exercise 3 in Section 3.

(Matlab) Let the controller K be given as

$$K = \left[\begin{array}{cccccc|cc} -4.69 & -1.28 & -1.02 & 7.29 & -0.937 & -0.736 & 5.69 & 1.6 \\ -2.42 & -3.29 & -3.42 & 6.53 & -1.17 & -0.922 & 2.42 & 4.12 \\ -1.48 & 0.418 & -0.666 & 2.52 & 0 & 0 & 1.48 & 1.05 \\ 19 & -0.491 & 1.34 & -0.391 & -6.16 & -5.59 & 0.943 & -0.334 \\ 24.8 & 2.35 & 3.73 & -8.48 & -7.58 & -3.61 & -3.45 & -3.96 \\ 4.38 & 4.02 & 3.2 & -9.41 & 2.54 & 0 & -4.38 & -5.03 \\ \hline 20 & -0.757 & 1.13 & -1 & -6.16 & -5.59 & 0 & 0 \end{array}\right].$$

Suppose the component G_1 is actually given by

$$G_1 + W_1\Delta \quad \text{with} \quad W_1(s) = \frac{s-2}{s+2}.$$

For the above controller, determine the largest r such that the loop remains stable for all Δ with

$$\Delta \in \mathbb{R}, \ |\Delta| < r \quad \text{or} \quad \Delta \in RH_\infty, \ \|\Delta\|_\infty < r.$$

Argue in terms of the Nyquist or Bode plot seen by the uncertainty Δ. Construct for both cases destabilizing perturbations of smallest size, and check that they lead, indeed, to an unstable controlled interconnection as expected.

a) (Matlab) Use the same data as in the previous exercise. Suppose now that the *pole* of G_2 is uncertain:

$$G_2(s) = \frac{1}{s-p}, \quad p \in \mathbb{R},\ |p-1| < r.$$

What is the largest possible r such that K still stabilizes P for all possible real poles p. What happens if the pole variation is allowed to be complex?

b) (Matlab) With W_1 as above, let G_1 be perturbed as

$$G_1 + W_1\Delta, \quad \Delta \in RH_\infty,\ \|\Delta\|_\infty < r_1$$

and let G_2 be given as

$$G_2(s) = \frac{1}{s-p}, \quad p \in \mathbb{R}, \quad |p-1| < r_2.$$

Find the largest $r = r_1 = r_2$ such that K still robustly stabilizes the system in face of these uncertainties. Plot a trade-off curve: For each r_1 in a suitable interval (which?), compute the largest $r_2(r_1)$ for which K is still robustly stabilizing and plot the graph of $r_2(r_1)$; comment!

2) Suppose $\mathbf{\Delta_c}$ is the set of all matrices with $\|\Delta\| < 1$ that have the structure

$$\Delta = \text{diag}(\delta_1, \dots, \delta_m) \in \mathbb{R}^{m\times m},$$

and consider the rationally perturbed matrix $A_\Delta = A + B\Delta(I - D\Delta)^{-1}C$ for real matrices A, B, C, D of compatible size. Derive a μ-test for the following properties:

a) Both $I - D\Delta$ and A_Δ are non-singular for all $\Delta \in \mathbf{\Delta_c}$.

b) For all $\Delta \in \mathbf{\Delta_c}$, $I - D\Delta$ is non-singular and A_Δ has all its eigenvalues in the open left-half plane.

c) For all $\Delta \in \mathbf{\Delta_c}$, $I - D\Delta$ is non-singular and A_Δ has all its eigenvalues in the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

3) Let $\mathbf{\Delta_c}$ be the same as in Exercise 2. For real vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^m$, give a formula for $\mu_{\mathbf{\Delta_c}}(ab^T)$. (SSV of rank one matrices.)

4) a) For the data

$$M_c = \begin{pmatrix} 12 & -3 & 2 \\ -1 & 7 & 8 \\ 5 & 3 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Delta_c} = \{\Delta_c = \begin{pmatrix} \Delta_{11} & \Delta_{12} & 0 \\ \Delta_{21} & \Delta_{22} & 0 \\ 0 & 0 & \Delta_{33} \end{pmatrix} \in \mathbb{C}^{3\times 3} \mid \|\Delta_c\| < 1\}$$

compute $\mu_{\boldsymbol{\Delta_c}}(M_c)$ with a Matlab m-file. You are allowed to use only the functions `max`, `eig` and a `for`-loop; in particular, don't use `mu`.

b) Let $M_c = \begin{pmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{pmatrix}$ and let $\boldsymbol{\Delta_c}$ be the set of all $\Delta_c = \mathrm{diag}(\Delta_1, \Delta_2)$ with full square blocks Δ_j satisfying $\|\Delta_j\| < 1$. Give a formula for the value

$$d_* = \min_{d>0} \left\| \begin{pmatrix} M_1 & dM_{12} \\ \frac{1}{d}M_{21} & M_2 \end{pmatrix} \right\|_2 .$$

where $\|M\|_2^2 = \mathrm{trace}(M^*M)$. How does d_* lead to an upper bound of $\mu_{\boldsymbol{\Delta}_c}(M)$? (Matlab) Compare this bound with the exact value in the previous exercise.

5 Nominal Performance Specifications

In our general scenario (Figure 21), we have collected various signals into the generalized disturbance w and the controlled variable z, and we assume that these signals are chosen to characterize the performance properties to be achieved by the controller. So far, however, we only included the requirement that K should render $z = S(P, K)w$ stable.

For any stabilizing controller, one can of course just directly investigate the transfer function elements of $S(P, K)$ and decide whether they are satisfactory or not. Most often, this just means that the Bode plots of these transfer functions should admit a desired shape that is dictated by the interpretation of the underlying signals. In this context you should remember the desired shapes for the sensitivity and the complementary sensitivity transfer matrices in a standard tracking problem.

For the purpose of analysis, a direct inspection of the closed-loop transfer matrix is no problem at all. However, if the interconnection is affected by uncertainties and if one would like to verify *robust performance*, or if one wishes to *design a controller*, it is required to translate the desired performance specifications into a weighted H_∞-norm criterion.

5.1 An Alternative Interpretation of the H_∞ Norm

We have seen in Section 1.2 that the H_∞-norm of a proper and stable G is just the energy gain of the corresponding LTI system.

Most often, reference or disturbance signals are persistent and can be assumed to be sinusoids. Such a signal is given as

$$w(t) = w_0 e^{i\omega_0 t} \tag{5.1}$$

with $w_0 \in \mathbb{C}^n$ and $\omega_0 \in \mathbb{R}$. If $\omega_0 = 0$ and $w_0 \in \mathbb{R}^n$, this defines the step function $w(t) = w_0$ of height w_0. (We note that this class of complex valued signals includes the set of all real-valued sinusoids. We work with this enlarged class to simplify the notation in the following arguments.) Let us choose as a measure of size for (5.1) the Euclidean norm of its amplitude:

$$\|w(.)\|_{\mathrm{RMS}} := \|w_0\|.$$

As indicated, this defines indeed a norm on the vector space of all sinusoids.

Let $w(.)$ as defined by (5.1) pass the LTI system defined by the proper and stable transfer matrix G to obtain $z(.)$. As well-known, $\lim_{t\to\infty}[z(t) - G(i\omega_0)w_0 e^{i\omega_0 t}] = 0$ such that the *steady-state response* is

$$(Gw)_s(t) = G(i\omega_0)w_0 e^{i\omega_0 t}.$$

(The subscript means that we only consider the steady-state response of Gw.) We infer

$$\|(Gw)_s\|_{\text{RMS}} = \|G(i\omega_0)w_0\|$$

and hence, due to $\|G(i\omega_0)w_0\| \leq \|G(i\omega)\|\|w_0\|$,

$$\frac{\|(Gw)_s\|_{\text{RMS}}}{\|w\|_{\text{RMS}}} \leq \|G(i\omega_0)\| \leq \|G\|_\infty.$$

Hence the gain of $w \to (Gw)_s$ is bounded by $\|G\|_\infty$. The gain actually turns out to be equal to $\|G\|_\infty$.

Theorem 5.1 *Let G be proper and stable. Then*

$$\sup_{w \text{ a sinusoid with } \|w\|_{\text{RMS}}>0} \frac{\|(Gw)_s\|_{\text{RMS}}}{\|w\|_{\text{RMS}}} = \|G\|_\infty.$$

The proof is instructive since it shows how to *construct a signal that leads to the largest amplification if passed through the system.*

Proof. Pick the frequency $\omega_0 \in \mathbb{R} \cup \{\infty\}$ with $\|G(i\omega_0)\| = \|G\|_\infty$.

Let us first assume that ω_0 is finite. Then take $w_0 \neq 0$ with $\|G(i\omega_0)w_0\| = \|G(i\omega_0)\|\|w_0\|$. (Direction of largest gain of $G(i\omega_0)$.) For the signal $w(t) = w_0 e^{i\omega_0 t}$ we infer

$$\frac{\|(Gw)_s\|_{\text{RMS}}}{\|w\|_{\text{RMS}}} = \frac{\|G(i\omega_0)w_0\|}{\|w_0\|} = \frac{\|G(i\omega_0)\|\|w_0\|}{\|w_0\|} = \|G(i\omega_0)\| = \|G\|_\infty.$$

Hence the gain $\frac{\|(Gw)_s\|_{\text{RMS}}}{\|w\|_{\text{RMS}}}$ for this signal is largest possible.

If ω_0 is infinite, take any sequence $\omega_j \in \mathbb{R}$ with $\omega_j \to \infty$, and construct at each ω_j the signal $w_j(.)$ as before. We infer

$$\frac{\|(Gw_j)_s\|_{\text{RMS}}}{\|w_j\|_{\text{RMS}}} = \|G(i\omega_j)\|$$

and this converges to $\|G(i\omega_0)\| = \|G\|_\infty$. Hence we cannot find a signal for which the worst amplification is attained, but we can come arbitrarily close.

■

As a generalization, sums of sinusoids are given as

$$w(t) = \sum_{j=1}^{N} w_j e^{i\omega_j t} \tag{5.2}$$

where N is the number of *pair-wise different* frequencies $\omega_j \in \mathbb{R}$, and $w_j \in \mathbb{C}^n$ is the complex amplitude at the frequency ω_j. As a measure of size for the signal (5.2) we employ

$$\|w\|_{\text{RMS}} := \sqrt{\sum_{j=1}^{N} \|w_j\|^2}.$$

Again, this defines a norm on the vector space of all sums of sinusoids. For any $w(.)$ defined by (5.2), the steady-state response is

$$(Gw)_s(t) = \sum_{j=1}^{N} G(i\omega_j) w_j e^{i\omega_j t}$$

and has norm

$$\|(Gw)_s\|_{\text{RMS}} = \sqrt{\sum_{j=1}^{N} \|G(i\omega_j)w_j\|^2}.$$

Again by $\|G(i\omega_j)w_j\| \leq \|G(i\omega_j)\|\|w_j\| \leq \|G\|_\infty \|w_j\|$, we infer

$$\|(Gw)_s\|_{\text{RMS}} \leq \|G\|_\infty \|w\|_{\text{RMS}}.$$

We arrive at the following generalization of the result given above.

Theorem 5.2 *Let G be proper and stable. Then*

$$\sup_{w \text{ a sum of sinusoids with } \|w\|_{\text{RMS}}>0} \frac{\|(Gw)_s\|_{\text{RMS}}}{\|w\|_{\text{RMS}}} = \|G\|_\infty.$$

Remark. We have separated the formulation of Theorem 5.1 from that of Theorem 5.2 in order to stress that $\frac{\|(Gw)_s\|_{\text{RMS}}}{\|w\|_{\text{RMS}}}$ can be rendered arbitrarily close to $\|G\|_\infty$ by using simple sinusoids as in (5.1); we do not require sums of sinusoids to achieve this approximation.

5.2 The Tracking Interconnection

Let us come back to the interconnection in Figure 17, and let K stabilize the interconnection.

5.2.1 Bound on Frequency Weighted System Gain

Often, performance specifications arise by specifying how signals have to be attenuated in the interconnection.

Typically, the reference r and the disturbance d are most significant at low frequencies. With real-rational proper and stable low-pass filters W_r, W_d, we hence assume that r, d are given as

$$r = W_r \tilde{r}, \quad d = W_d \tilde{d}$$

where $\tilde{r}(.)$, $\tilde{d}(.)$ are sinusoids or sums of sinusoids. Similarly, the measurement noise n is most significant at high frequencies. With a real-rational proper and stable high-pass filter W_n, we hence assume that n is given as

$$n = W_n \tilde{n}$$

where $\tilde{n}(.)$ is a sum of sinusoids. Finally, the size of the unfiltered signals is assumed to be bounded as

$$\left\| \begin{pmatrix} \tilde{d} \\ \tilde{n} \\ \tilde{r} \end{pmatrix} \right\|_{\mathrm{RMS}}^2 = \|\tilde{r}\|_{\mathrm{RMS}}^2 + \|\tilde{d}\|_{\mathrm{RMS}}^2 + \|\tilde{n}\|_{\mathrm{RMS}}^2 \leq 1.$$

Remark. In our signal-based approach all signals are assumed to enter the interconnection together. Hence it is reasonable to bound the stacked signal instead of working with individual bounds on $\|\tilde{r}\|_{\mathrm{RMS}}$, $\|\tilde{d}\|_{\mathrm{RMS}}$, $\|\tilde{n}\|_{\mathrm{RMS}}$. Recall, however, that the above inequality implies

$$\|\tilde{r}\|_{\mathrm{RMS}}^2 \leq 1, \quad \|\tilde{d}\|_{\mathrm{RMS}}^2 \leq 1, \quad \|\tilde{n}\|_{\mathrm{RMS}}^2 \leq 1,$$

and that it is implied by

$$\|\tilde{r}\|_{\mathrm{RMS}}^2 \leq \frac{1}{3}, \quad \|\tilde{d}\|_{\mathrm{RMS}}^2 \leq \frac{1}{3}, \quad \|\tilde{n}\|_{\mathrm{RMS}}^2 \leq \frac{1}{3}.$$

The goal is to keep the norm of the steady-state error e_s small, no matter which of these signals enters the interconnection. If we intend to achieve $\|e_s\|_{\mathrm{RMS}} \leq \epsilon$, we can as well rewrite this condition with $W_e := \frac{1}{\epsilon}$ as $\|\tilde{e}_s\|_{\mathrm{RMS}} \leq 1$ for

$$\tilde{e} = W_e e.$$

To proceed to the general framework, let us introduce

$$z = e, \;\; \tilde{z} = \tilde{e} \;\; \text{and} \;\; w = \begin{pmatrix} d \\ n \\ r \end{pmatrix}, \;\; \tilde{w} = \begin{pmatrix} \tilde{d} \\ \tilde{n} \\ \tilde{r} \end{pmatrix}$$

as well as the weightings

$$W_z = W_e \;\; \text{and} \;\; W_w = \begin{pmatrix} W_d & 0 & 0 \\ 0 & W_n & 0 \\ 0 & 0 & W_r \end{pmatrix}.$$

In the general framework, the original closed-loop interconnection was described as

$$z = S(P, K)w.$$

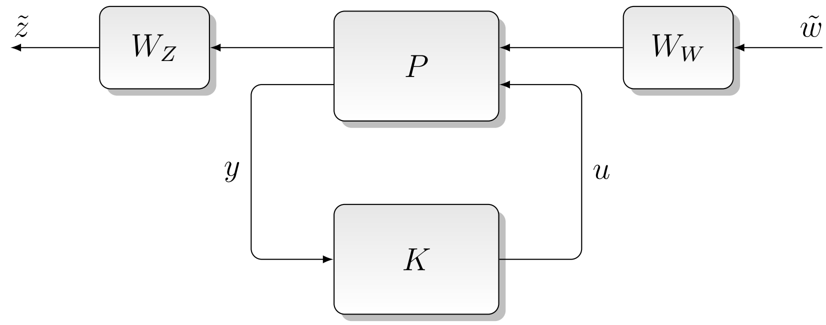


Figure 50: Weighted closed-loop interconnection

Since the desired performance specification is formulated in terms of $\tilde{z}$ and $\tilde{w}$, we introduce these signals with

$$\tilde{z} = W_z z \quad \text{and} \quad w = W_w \tilde{w}$$

to get the weighted closed-loop interconnection (Figure 50)

$$\tilde{z} = [W_z S(P, K) W_w]\tilde{w}.$$

Recall that the desired performance specification was reduced to the fact that

$$\|\tilde{w}\|_{\text{RMS}} \leq 1 \quad \text{implies} \quad \|\tilde{z}_s\|_{\text{RMS}} \leq 1.$$

By Theorem 5.2, this requirement is *equivalent* to

$$\|W_z S(P, K) W_w\|_\infty \leq 1. \tag{5.3}$$

We have arrived at those performance specifications that can be handled with the techniques developed in these notes: Bounds on the weighted H_∞-norm of the performance channels.

Let us recall that this specification is equivalent to

$$\|W_z(i\omega) S(P, K)(i\omega) W_w(i\omega)\| \leq 1 \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}.$$

This reveals the following two interpretations:

- **Loop-shape interpretation**. The shape of the frequency response $\omega \to S(P, K)(i\omega)$ is compatible with the requirement that the maximal singular value of the weighted frequency response $\omega \to W_z(i\omega) S(P, K)(i\omega) W_w(i\omega)$ does not exceed one. Roughly, this amounts to bounding all singular values of $S(P, K)(i\omega)$ from above with a bound that varies according the variations of the singular values of $W_z(i\omega)$ and $W_w(i\omega)$. This rough interpretation is accurate if $W_z(i\omega)$ and $W_w(i\omega)$ are just scalar valued since (5.3) then just amounts to

$$\|S(P, K)(i\omega)\| \leq \frac{1}{|W_z(i\omega) W_w(i\omega)|} \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}.$$

It is important to note that one cannot easily impose a lower bound on the smallest singular value of $S(P,K)(i\omega)$! Instead, desired minimal amplifications are enforced by imposing upper bounds on the largest singular value of 'complementary' transfer functions - for that purpose one should recall the interplay of sensitivity and complementary sensitivity matrices.

- **Disturbance attenuation interpretation**. For all disturbances that are defined through

$$w(t) = \sum_{j=1}^{N} W(i\omega_j) w_j e^{i\omega_j t} \quad \text{with} \quad \sum_{j=1}^{N} \|w_j\|^2 \leq 1,$$

(5.3) implies that the steady-state response z_s of $z = S(P,K)w$ satisfies

$$z_s(t) = \sum_{j=1}^{N} z_j e^{i\omega_j t} \quad \text{with} \quad \sum_{j=1}^{N} \|Z(i\omega_j) z_j\|^2 \leq 1.$$

For pure sinusoids, any disturbance satisfying

$$w(t) = W(i\omega) w e^{i\omega t} \quad \text{with} \quad \|w\|^2 \leq 1$$

leads to a steady-state response

$$z_s(t) = z e^{i\omega t} \quad \text{with} \quad \|Z(i\omega) z\|^2 \leq 1.$$

This leads to a very clear interpretation of *matrix valued weightings*: Sinusoids of frequency ω with an amplitude in the ellipsoid $\{W_w(i\omega)w \mid \|w\| \leq 1\}$ lead to a steady-state sinusoidal response with amplitude in the ellipsoid $\{z \mid \|W_z(i\omega)z\| \leq 1\}$. Hence W_w defines the ellipsoid which captures the a priori knowledge of the amplitudes of the incoming disturbances and W_z defines the ellipsoids that captures desired amplitudes of the controlled output. Through the use of matrix valued weightings one can hence enforce spatial effects, such as quenching the output error mainly in a certain direction.

Note that $W_z S(P,K) W_w$ is nothing but $S(\tilde{P}, K)$ for

$$\begin{pmatrix} \tilde{z} \\ y \end{pmatrix} = \tilde{P} \begin{pmatrix} \tilde{w} \\ u \end{pmatrix} = \begin{pmatrix} W_z P_{11} W_w & W_z P_{12} \\ P_{21} W_w & P_{22} \end{pmatrix} \begin{pmatrix} \tilde{w} \\ u \end{pmatrix}.$$

Instead of first closing the loop and then weighting the controlled system, one can as well first weight the open-loop interconnection and then close the loop.

5.2.2 Frequency Weighted Model Matching

In design, a typical specification is to let one or several transfer matrices in an interconnection come close to an ideal model. Let us suppose that the real-rational proper and

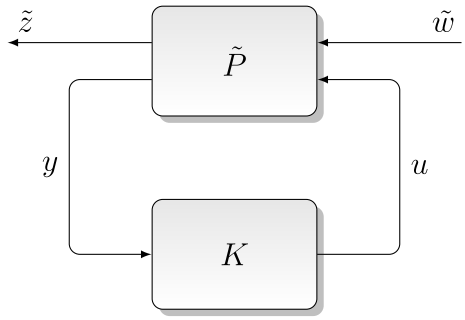


Figure 51: Weighted closed-loop interconnection

stable W_m is an ideal model. Moreover, the controller should render $S(P,K)$ to match this ideal model over certain frequency ranges. With suitable real-rational weightings W_1 and W_2, this amounts to render $\|W_1(i\omega)[S(P,K)(i\omega) - W_m(i\omega)]W_2(i\omega)\| \leq \gamma$ satisfied for all frequencies $\omega \in \mathbb{R} \cup \{\infty\}$ where γ is small. By incorporating the desired bound γ into the weightings (replace W_1 by $\frac{1}{\gamma}W_1$), we arrive at the performance specification

$$\|W_1[S(P,K) - W_m]W_2\|_\infty \leq 1. \tag{5.4}$$

To render this inequality satisfied, one tries 'shape the closed-loop frequency response by pushing it towards a desired model'. In this fashion, one can incorporate for each transfer function of $S(P,K)$ both amplitude and phase specifications. If there is no question about which ideal model W_m to take, this is the method of choice.

Again, we observe that this performance specification can be rewritten as

$$\|S(\tilde{P},K)\|_\infty \leq 1$$

where $\tilde{P}$ is defined as

$$\tilde{P} = \begin{pmatrix} W_1[P_{11} - W_m]W_2 & W_1P_{12} \\ P_{21}W_2 & P_{22} \end{pmatrix}.$$

Remark. Note that the choice $W_m = 0$ of the ideal model leads back to imposing a direct bound on the system gain as discussed before.

In summary, typical signal based performance specifications can be re-formulated as a general frequency weighted model-matching requirement which leads to a *bound on the H_∞-norm of the matrix-weighted closed-loop transfer matrix.*

5.3 The General Paradigm

Starting from

$$\begin{pmatrix} z \\ y \end{pmatrix} = P\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix}, \tag{5.5}$$

we have seen how to translate the two most important performance specifications on the closed-loop system

$$S(P,K),$$

weighted gain-bounds and weighted model-matching, into the specification

$$\|S(\tilde{P},K)\|_\infty \leq 1 \tag{5.6}$$

for the weighted open-loop interconnection

$$\begin{pmatrix} \tilde{z} \\ y \end{pmatrix} = \tilde{P} \begin{pmatrix} \tilde{w} \\ u \end{pmatrix} = \begin{pmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \tilde{w} \\ u \end{pmatrix}. \tag{5.7}$$

So far, we have largely neglected any technical hypotheses on the weighting matrices that are involved in building $\tilde{P}$ from P. In fact, any controller to be considered should stabilize both P and $\tilde{P}$. Hence we have to require that both interconnections define generalized plants, and these are the only properties to be obeyed by any weighting matrices that are incorporated in the interconnection.

Hypothesis 5.3 *The open-loop interconnections (5.5) and (5.7) are generalized plants.*

Note that P and $\tilde{P}$ have the same lower right block P_{22}. This is the reason why any controller that stabilizes P also stabilizes $\tilde{P}$, and vice versa.

Lemma 5.4 *Let P and $\tilde{P}$ be generalized plants. A controller K stabilizes P if and only if K stabilizes $\tilde{P}$.*

Proof. If K stabilizes P, then K stabilizes P_{22}. Since $\tilde{P}$ is a generalized plant and has P_{22} as its right-lower block, K also stabilizes $\tilde{P}$. The converse follows by interchanging the role of P and $\tilde{P}$.

■

Hence the class of stabilizing controller for P and for $\tilde{P}$ are identical.

From now on we assume that all performance weightings are already incorporated in P. Hence the performance specification is given by $\|S(P,K)\|_\infty \leq 1$.

Remark. In practical controller design, it is often important to keep P and $\tilde{P}$ separated. Indeed, the controller will be *designed* on the basis of $\tilde{P}$ to obey $\|S(\tilde{P},K)\|_\infty \leq 1$, but then it is often much more instructive to directly investigate the unweighted frequency response $\omega \to S(P,K)(i\omega)$ in order to judge whether the designed controller leads to the desired closed-loop specifications.

6 Robust Performance Analysis

6.1 Problem Formulation

To test robust performance, we proceed as for robust stability: We identify the performance signals, we pull out the uncertainties and introduce suitable weightings for the uncertainties such that we arrive at the framework as described in Section 4.5. Moreover, we incorporate in this framework the performance weightings as discussed in Section 5 to reduce the desired performance specification to an H_∞ norm bound on the performance channel. In Figure 52 we have displayed the resulting open-loop interconnection, the interconnection if closing the loop as $u = Ky$, and the controlled interconnection with uncertainty.

We end up with the controlled uncertain system as described by

$$\begin{pmatrix} z_\Delta \\ z \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix}, \ u = Ky, \ w_\Delta = \Delta z_\Delta, \ \Delta \in \mathbf{\Delta}.$$

Let us now formulate the precise hypotheses on P, on the uncertainty class Δ, and on the performance specification as follows.

Hypothesis 6.1

- *P is a generalized plant.*
- *The set of uncertainties is given as*

$$\mathbf{\Delta} := \{\Delta \in RH_\infty \mid \Delta(i\omega) \in \mathbf{\Delta}_c \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}\}$$

  where $\mathbf{\Delta}_c$ is the set of all matrices Δ_c structured as (4.46) and satisfying $\|\Delta_c\| < 1$.
- *The direct feed-through P_{11} and $\mathbf{\Delta}_c$ are such that*

$$I - P_{11}(\infty)\Delta_c \ \text{ is non-singular for all } \ \Delta_c \in \mathbf{\Delta}_c.$$

- *The performance of the system is as desired if the H_∞-norm of the channel $w \to z$ is smaller than one.*

We use the the brief notation

$$P_\Delta = S(\Delta, P) = \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix} + \begin{pmatrix} P_{21} \\ P_{32} \end{pmatrix} \Delta (I - P_{11}\Delta)^{-1} \begin{pmatrix} P_{12} & P_{13} \end{pmatrix}$$

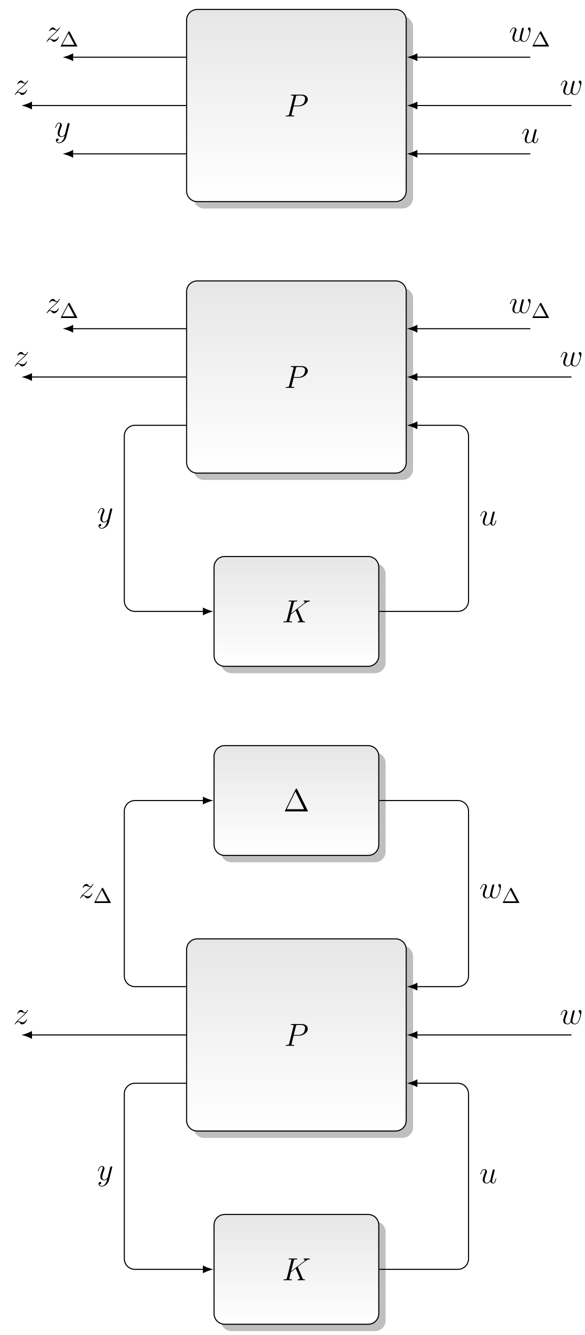


Figure 52: Open-loop interconnection, controlled interconnection, uncertain controlled interconnection.

for the perturbed open-loop interconnection. Then the unperturbed open-loop interconnection is nothing but

$$P_0 = S(0, P) = \begin{pmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{pmatrix}.$$

Suppose that K stabilizes P. Then the perturbed and unperturbed controlled interconnections are described by

$$z = S(P_\Delta, K)w \quad \text{and} \quad z = S(P_0, K)w$$

respectively. If K stabilizes P and if it leads to

$$\|S(P_0, K)\|_\infty \leq 1,$$

we say that K achieves **nominal performance** for P.

Accordingly, we can formulate the corresponding analysis and synthesis problems.

Nominal Performance Analysis Problem
For a given fixed controller K, test whether it achieves nominal performance for P.

Nominal Performance Synthesis Problem
Find a controller K that achieves nominal performance for P.

The analysis problem is very easy to solve: Check whether K stabilizes P, and plot $\omega \to \|S(P_0, K)(i\omega)\|$ in order to verify whether this value remains smaller than 1. Note that H_∞-norm bounds of this sort can be verified much more efficiently on the basis of a state-space test, as will be discussed in Section 7.2.

The synthesis problem amounts to finding a stabilizing controller K for P that renders the H_∞-norm $\|S(P_0, K)\|_\infty$ smaller than 1. This is the celebrated H_∞-control problem and will be discussed in Section 7.4.

The main subject of this section is robust performance analysis. If

$$K \text{ stabilizes } P_\Delta = S(\Delta, P) \text{ and } \|S(P_\Delta, K)\|_\infty \leq 1 \text{ for all } \Delta \in \mathbf{\Delta},$$

we say that

$$K \text{ achieves } \textbf{robust performance} \text{ for } S(\Delta, P) \text{ against } \mathbf{\Delta}.$$

Let us again formulate the related analysis and synthesis problems explicitly.

Robust Performance Analysis Problem
For a given fixed controller K, test whether it achieves robust performance for $S(\Delta, P)$ against $\mathbf{\Delta}$.

Robust Performance Synthesis Problem
Find a controller K that achieves robust performance for $S(\Delta, P)$ against $\mathbf{\Delta}$.

6.2 Testing Robust Performance

Let us assume throughout that K stabilizes P what implies that $N := S(P, K)$ is stable. Introduce the partition

$$\begin{pmatrix} z_\Delta \\ z \end{pmatrix} = S(P,K) \begin{pmatrix} w_\Delta \\ w \end{pmatrix} = N \begin{pmatrix} w_\Delta \\ w \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w \end{pmatrix}.$$

Then we infer

$$S(P_\Delta, K) = S(\Delta, N) = N_{22} + N_{21}\Delta(I - M\Delta)^{-1}N_{12}.$$

Hence, K achieves robust performance if the robust stability condition

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}$$

or equivalently

$$\det(I - M(i\omega)\Delta_c) \neq 0 \ \text{ for all } \ \Delta_c \in \boldsymbol{\Delta_c}, \ \omega \in \mathbb{R} \cup \{\infty\}$$

and the performance bound

$$\|N_{22} + N_{21}\Delta(I - M\Delta)^{-1}N_{12}\| \leq 1 \ \text{ for all } \ \Delta \in \boldsymbol{\Delta}$$

or equivalently

$$\|N_{22}(i\omega) + N_{21}(i\omega)\Delta(i\omega)(I - M(i\omega)\Delta(i\omega))^{-1}N_{12}(i\omega)\| \leq 1 \quad \text{for all } \ \Delta \in \boldsymbol{\Delta}, \ \omega \in \mathbb{R} \cup \{\infty\}$$

or equivalently

$$\|N_{22}(i\omega) + N_{21}(i\omega)\Delta_c(I - M(i\omega)\Delta_c)^{-1}N_{12}(i\omega)\| \leq 1 \ \text{ for all } \ \Delta_c \in \boldsymbol{\Delta_c}, \ \omega \in \mathbb{R} \cup \{\infty\}$$

hold true.

Similarly as for robust stability, for a fixed frequency we arrive at a problem in linear algebra which is treated in the next section.

6.3 The Main Loop Theorem

Here is the linear algebra problem that needs to be investigated: Given the set $\boldsymbol{\Delta_c}$ and the complex matrix

$$N_c = \begin{pmatrix} M_c & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \quad \text{with} \ \ N_{22} \ \text{ of size } \ q_2 \times p_2,$$

test whether the following two conditions hold:

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ and } \ \|N_{22} + N_{21}\Delta_c(I - M_c\Delta_c)^{-1}N_{12}\| \leq 1 \ \text{ for all } \ \Delta_c \in \boldsymbol{\Delta_c}.$$

Here is the fundamental **trick** to solve this problem: The condition $\|N_{22} + N_{21}\Delta_c(I - M_c\Delta_c)^{-1}N_{12}\| = \|S(\Delta_c, N_c)\| \leq 1$ is *equivalent* to

$$\det(I - S(\Delta_c, N_c)\widehat{\Delta}_c) \neq 0 \ \text{ for all } \ \widehat{\Delta}_c \in \mathbb{C}^{p_2 \times q_2}, \ \|\widehat{\Delta}_c\| < 1.$$

We just need to recall that the SSV of a complex matrix equals its norm if the uncertainty structure just consists of one full block.

Let us hence define

$$\widehat{\boldsymbol{\Delta}}_{\boldsymbol{c}} = \{\widehat{\Delta}_c \in \mathbb{C}^{p_2 \times q_2} \mid \|\widehat{\Delta}_c\| < 1\}.$$

We infer that, for all $\Delta_c \in \boldsymbol{\Delta_c}$,

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ and } \ \|S(\Delta_c, N_c)\| \leq 1$$

if and only if, for all $\Delta_c \in \boldsymbol{\Delta_c}$ and $\widehat{\Delta}_c \in \widehat{\boldsymbol{\Delta}}_{\boldsymbol{c}}$,

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ and } \ \det(I - S(\Delta_c, N_c)\widehat{\Delta}_c) \neq 0$$

if and only if, for all $\Delta_c \in \boldsymbol{\Delta_c}$ and $\widehat{\Delta}_c \in \widehat{\boldsymbol{\Delta}}_{\boldsymbol{c}}$,

$$\det \begin{pmatrix} I - M_c\Delta_c & -N_{12}\widehat{\Delta}_c \\ -N_{21}\Delta_c & I - N_{22}\widehat{\Delta}_c \end{pmatrix} \neq 0$$

if and only if, for all $\Delta_c \in \boldsymbol{\Delta_c}$ and $\widehat{\Delta}_c \in \widehat{\boldsymbol{\Delta}}_{\boldsymbol{c}}$,

$$\det \left(I - \begin{pmatrix} M_c & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} \Delta_c & 0 \\ 0 & \widehat{\Delta}_c \end{pmatrix} \right) \neq 0.$$

Note that we have used in these derivation the following simple consequence of the well-known Schur formula for the determinant:

$$\det(I - S(\Delta_c, N_c)\widehat{\Delta}_c) = \det(I - [N_{22} + N_{21}\Delta_c(I - M_c\Delta_c)^{-1}N_{12}]\widehat{\Delta}_c) =$$

$$= \det([I - N_{22}\widehat{\Delta}_c] - [N_{21}\Delta_c](I - M_c\Delta_c)^{-1}[N_{12}\widehat{\Delta}_c)]) = \det \begin{pmatrix} I - M_c\Delta_c & -N_{12}\widehat{\Delta}_c \\ -N_{21}\Delta_c & I - N_{22}\widehat{\Delta}_c \end{pmatrix}.$$

This motivates to introduce the set of **extended** matrices

$$\boldsymbol{\Delta_e} := \left\{ \begin{pmatrix} \Delta_c & 0 \\ 0 & \widehat{\Delta}_c \end{pmatrix} : \ \Delta_c \in \boldsymbol{\Delta_c}, \ \widehat{\Delta}_c \in \mathbb{C}^{p_2 \times q_2}, \ \|\widehat{\Delta}_c\| < 1 \right\}$$

which consists of adjoining to the original structure one full block. We have proved the following **Main Loop Theorem**.

Theorem 6.2 *The two conditions*

$$\mu_{\boldsymbol{\Delta_c}}(M_c) \leq 1 \quad \textit{and} \quad \|S(\Delta_c, N_c)\| \leq 1 \quad \textit{for all} \quad \Delta_c \in \boldsymbol{\Delta_c}$$

are equivalent to

$$\mu_{\boldsymbol{\Delta}_e}(N_c) \leq 1.$$

This result reduces the desired condition to just another SSV-test on the matrix N_c with respect to the extended structure $\boldsymbol{\Delta_e}$.

Typically, a computation of $\mu_{\boldsymbol{\Delta}_e}(N_c)$ will lead to an inequality

$$\mu_{\boldsymbol{\Delta}_e}(N_c) \leq \gamma$$

with a bound $\gamma > 0$ different from one. The consequences that can then be drawn can be easily obtained by re-scaling. In fact, this inequality leads to

$$\mu_{\boldsymbol{\Delta}_e}(\frac{1}{\gamma}N_c) \leq 1.$$

This is equivalent to

$$\mu_{\boldsymbol{\Delta_c}}(\frac{1}{\gamma}M_c) \leq 1$$

and

$$\|\frac{1}{\gamma}N_{22} + \frac{1}{\gamma}N_{21}\Delta_c(I - \frac{1}{\gamma}M_c\Delta_c)^{-1}\frac{1}{\gamma}N_{12}\| \leq 1 \quad \text{for all} \quad \Delta_c \in \boldsymbol{\Delta_c}.$$

Both conditions are clearly nothing but

$$\mu_{\boldsymbol{\Delta_c}}(M_c) \leq \gamma$$

and

$$\|N_{22} + N_{21}[\frac{1}{\gamma}\Delta_c](I - M_c[\frac{1}{\gamma}\Delta_c])^{-1}N_{12}\| \leq \gamma \quad \text{for all} \quad \Delta_c \in \boldsymbol{\Delta_c}.$$

We arrive at

$$\det(I - M_c\Delta_c) \neq 0 \quad \text{for all} \quad \Delta_c \in \frac{1}{\gamma}\boldsymbol{\Delta_c}$$

and

$$\|N_{22} + N_{21}\Delta_c(I - M_c\Delta_c)^{-1}N_{12}\| \leq \gamma \quad \text{for all} \quad \Delta_c \in \frac{1}{\gamma}\boldsymbol{\Delta_c}.$$

Hence a general bound γ different from one leads to non-singularity conditions and a performance bound γ for the class of complex matrices $\frac{1}{\gamma}\boldsymbol{\Delta_c}$.

A more general scaling result that is proved analogously can be formulated as follows.

Lemma 6.3 *The scaled SSV-inequality*

$$\mu_{\boldsymbol{\Delta_e}}\left(N_c\begin{pmatrix}\gamma_1 I & 0\\ 0 & \gamma_2 I\end{pmatrix}\right) \leq \gamma_3$$

is equivalent to

$$\det(I - M_c\Delta_c) \neq 0 \quad \textit{for all} \quad \Delta_c \in \frac{\gamma_1}{\gamma_3}\boldsymbol{\Delta_c}$$

and

$$\|S(\Delta_c, N_c)\| \leq \frac{\gamma_3}{\gamma_2} \quad \textit{for all} \quad \Delta_c \in \frac{\gamma_1}{\gamma_3}\boldsymbol{\Delta_c}.$$

This result allows to investigate the trade-off between the size of the uncertainty and the worst possible norm $\|S(\Delta_c, N_c)\|$ by varying γ_1, γ_2 and computing the SSV giving the bound γ_3.

Note that we can as well draw conclusions of the following sort: If one wishes to guarantee

$$\det(I - M_c\Delta_c) \neq 0 \ \text{ and } \ \|S(\Delta_c, N_c)\| \leq \beta \ \text{ for all } \ \Delta_c \in \alpha\boldsymbol{\Delta_c}$$

for some bounds $\alpha > 0$, $\beta > 0$, one needs to perform the SSV-test

$$\mu_{\boldsymbol{\Delta_e}}\left(N_c\begin{pmatrix}\alpha I & 0\\ 0 & \frac{1}{\beta} I\end{pmatrix}\right) \leq 1.$$

6.4 The Main Robust Stability and Robust Performance Test

If we combine the findings of Section (6.2) with the main loop theorem, we obtain the following result.

Theorem 6.4 *Let* $N = \begin{pmatrix} M & N_{12}\\ N_{21} & N_{22}\end{pmatrix}$ *be a proper and stable transfer matrix. For all* $\Delta \in \boldsymbol{\Delta}$,

$$(I - M\Delta)^{-1} \in RH_\infty \quad \textit{and} \quad \|S(\Delta, N)\|_\infty \leq 1$$

if and only if

$$\mu_{\boldsymbol{\Delta_e}}(N(i\omega)) \leq 1 \quad \textit{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}.$$

Combining all the insights we have gained so far leads us to the most fundamental result in SSV-theory, the test of robust stability and robust performance against structured uncertainties.

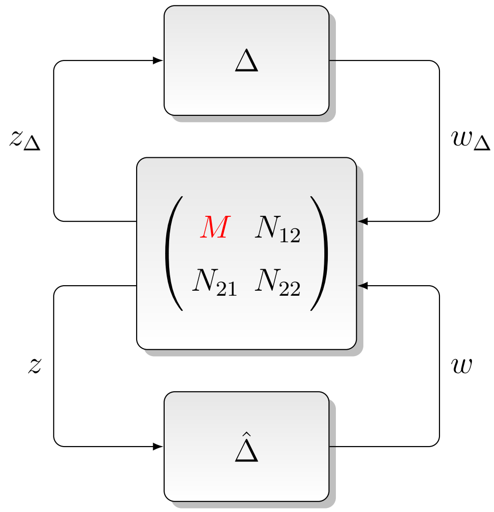


Figure 53: Equivalent Robust Stability Test

Corollary 6.5 *If K stabilizes P, and if*

$$\mu_{\boldsymbol{\Delta_e}}(N(i\omega)) \leq 1 \quad \textit{for all} \quad \omega \in \mathbb{R} \cup \{\infty\},$$

then K achieves robust performance for $S(\Delta, P)$ against all $\Delta \in \boldsymbol{\Delta}$.

At the outset, it looks more complicated to test robust performance if compared to robust stability. However, the main loop theorem implies that the test of robust performance is just another SSV test with respect to the extended block structure $\boldsymbol{\Delta_e}$.

Accidentally (and with no really deep consequence), the SSV-test for robust performance can be viewed as a robust stability test for the interconnection displayed in Figure 53.

6.5 Summary

Suppose that K stabilizes the generalized plant P and suppose that the controlled uncertain system is described as

$$\begin{pmatrix} z_\Delta \\ z \end{pmatrix} = S(P,K)\begin{pmatrix} w_\Delta \\ w \end{pmatrix} = N\begin{pmatrix} w_\Delta \\ w \end{pmatrix} = \begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} w_\Delta \\ w \end{pmatrix}, \quad w_\Delta = \Delta z_\Delta$$

with proper and stable Δ satisfying

$$\Delta(i\omega) \in \boldsymbol{\Delta_c} \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}.$$

Then the controller K achieves

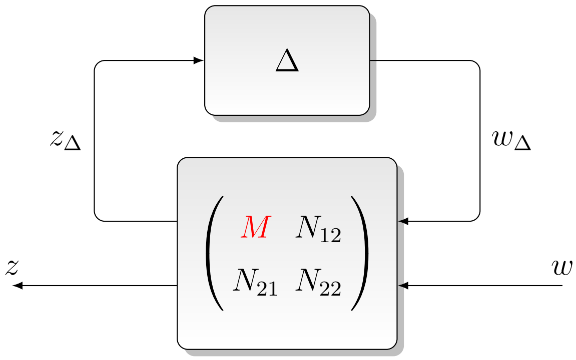


Figure 54: Summary

- Robust stability if

$$\mu_{\boldsymbol{\Delta_c}}(M(i\omega)) \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

- Nominal performance if

$$\|N_{22}(i\omega)\| \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

- Robust performance if

$$\mu_{\boldsymbol{\Delta_e}}(N(i\omega)) \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

In pictures, this can be summarized as follows. Robust stability is guaranteed by an SSV-test on left-upper block M of $N = S(P, K)$, nominal performance is guaranteed by an SV-test on the right-lower block N_{22} of $N = S(P, K)$, and robust performance is guaranteed by an SSV-test on the whole $N = S(P, K)$ with respect to the extended block structure.

6.6 An Example

Suppose some controlled system is described with

$$N(s) = \left(\begin{array}{ccc|c} \frac{1}{2s+1} & 1 & \frac{s-2}{2s+4} & \frac{s-0.1}{s+1} \\ -1 & \frac{s}{s^2+s+1} & \frac{1}{(s+1)^2} & 0.1 \\ \frac{3s}{s+5} & \frac{-1}{4s+1} & 1 & \frac{10}{s+4} \\ \hline \frac{1}{s+2} & \frac{0.1}{s^2+s+1} & \frac{s-1}{s+1} & 1 \end{array}\right).$$

Let $\boldsymbol{\Delta_c}$ be the set of Δ_c with $\|\Delta_c\| < 1$ and

$$\Delta_c = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \Delta_1 \in \mathbb{C}^{2\times 2}, \quad \Delta_2 \in \mathbb{C}.$$

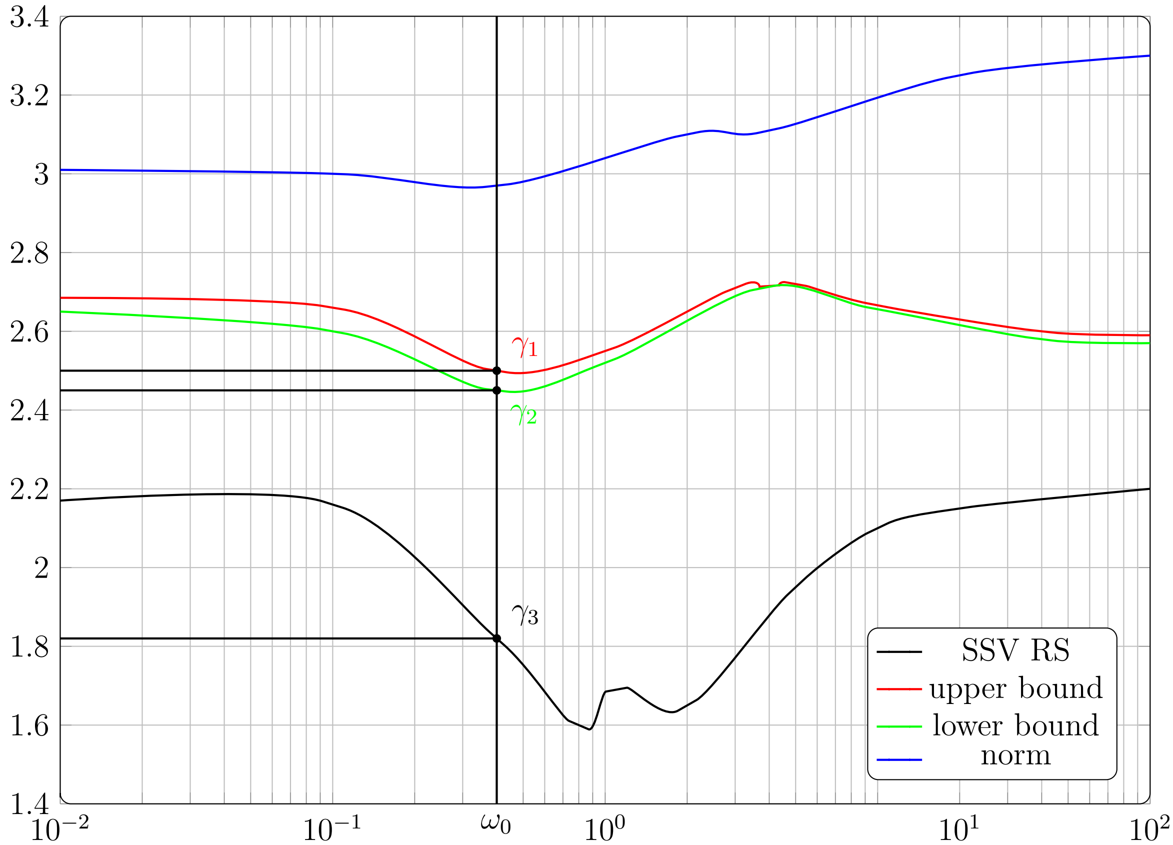


Figure 55: Norm of N, upper and lower bound on SSV of N, SSV of M.

The extended set $\boldsymbol{\Delta_e}$ consists of all Δ_e with $\|\Delta_e\| < 1$ and

$$\Delta_e = \left(\begin{array}{cc|c} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ \hline 0 & 0 & \widehat{\Delta} \end{array}\right), \ \Delta_1 \in \mathbb{C}^{2\times 2},\ \Delta_2 \in \mathbb{C},\ \widehat{\Delta} \in \mathbb{C}.$$

To test robust stability, we plot $\omega \to \mu_{\boldsymbol{\Delta_c}}(M(i\omega))$, to test nominal performance, we plot $\omega \to \|N_{22}(i\omega)\|$, and the robust performance test requires to plot $\omega \to \mu_{\boldsymbol{\Delta_e}}(N(i\omega))$.

Let us first look at a frequency-by-frequency interpretation of the SSV plot of N with respect to the extended structure (Figure 55).

With the upper bound, we infer $\mu_{\boldsymbol{\Delta_e}}(N(i\omega_0)) \leq \gamma_1$ what implies

$$\|S(\Delta_c, N(i\omega_0))\| \leq \gamma_1 \ \text{ for all } \ \Delta_c \in \frac{1}{\gamma_1}\boldsymbol{\Delta_c}.$$

At the frequency $i\omega_0$, one has a guaranteed performance level γ_1 for the uncertainty set $\frac{1}{\gamma_1}\boldsymbol{\Delta_c}$.

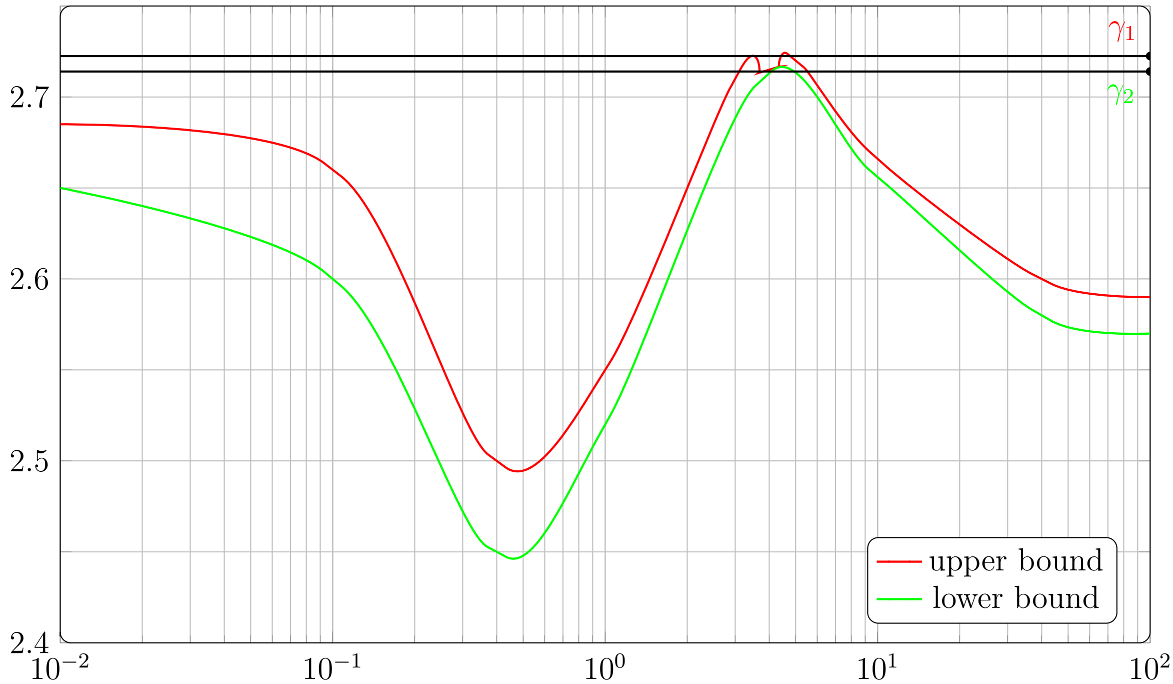


Figure 56: Upper and lower bound on SSV of N

With the lower bound, we infer $\mu_{\mathbf{\Delta}_e}(N(i\omega_0)) > \gamma_2$. This implies that

$$\det(I - M(i\omega_0)\Delta_c) = 0 \ \text{ for some } \ \Delta_c \in \frac{1}{\gamma_2}\boldsymbol{\Delta_c}$$

or

$$\|S(\Delta_c, N(i\omega_0))\| > \gamma_2 \ \text{ for some } \ \Delta_c \in \frac{1}{\gamma_2}\boldsymbol{\Delta_c}.$$

We can exploit the knowledge of the SSV curve for robust stability to exclude the first property due to $\gamma_3 < \gamma_2$. (Provide all arguments!) Hence, at this frequency we can violate the performance bound γ_2 by some matrix in the complex uncertainty set $\frac{1}{\gamma_2}\boldsymbol{\Delta_c}$.

Let us now interpret the upper and lower bound plots of the SSV of N for all frequencies (Figure 56).

Since the upper bound is not larger than 2.72 for all frequencies, we infer that

$$\|S(\Delta, N)\|_\infty \leq 2.72 \ \text{ for all } \ \Delta \in \frac{1}{2.72}\boldsymbol{\Delta} \approx 0.367\boldsymbol{\Delta}$$

Since the lower bound is larger that 2.71 for some frequency, we infer that either

$$(I - M\Delta)^{-1} \ \text{ is unstable for some } \ \Delta \in \frac{1}{2.71}\boldsymbol{\Delta} \approx 0.369\boldsymbol{\Delta}$$

or that

$$\|S(\Delta, N)\|_\infty > 2.71 \ \text{ for some } \ \Delta \in \frac{1}{2.71}\boldsymbol{\Delta} \approx 0.369\boldsymbol{\Delta}.$$

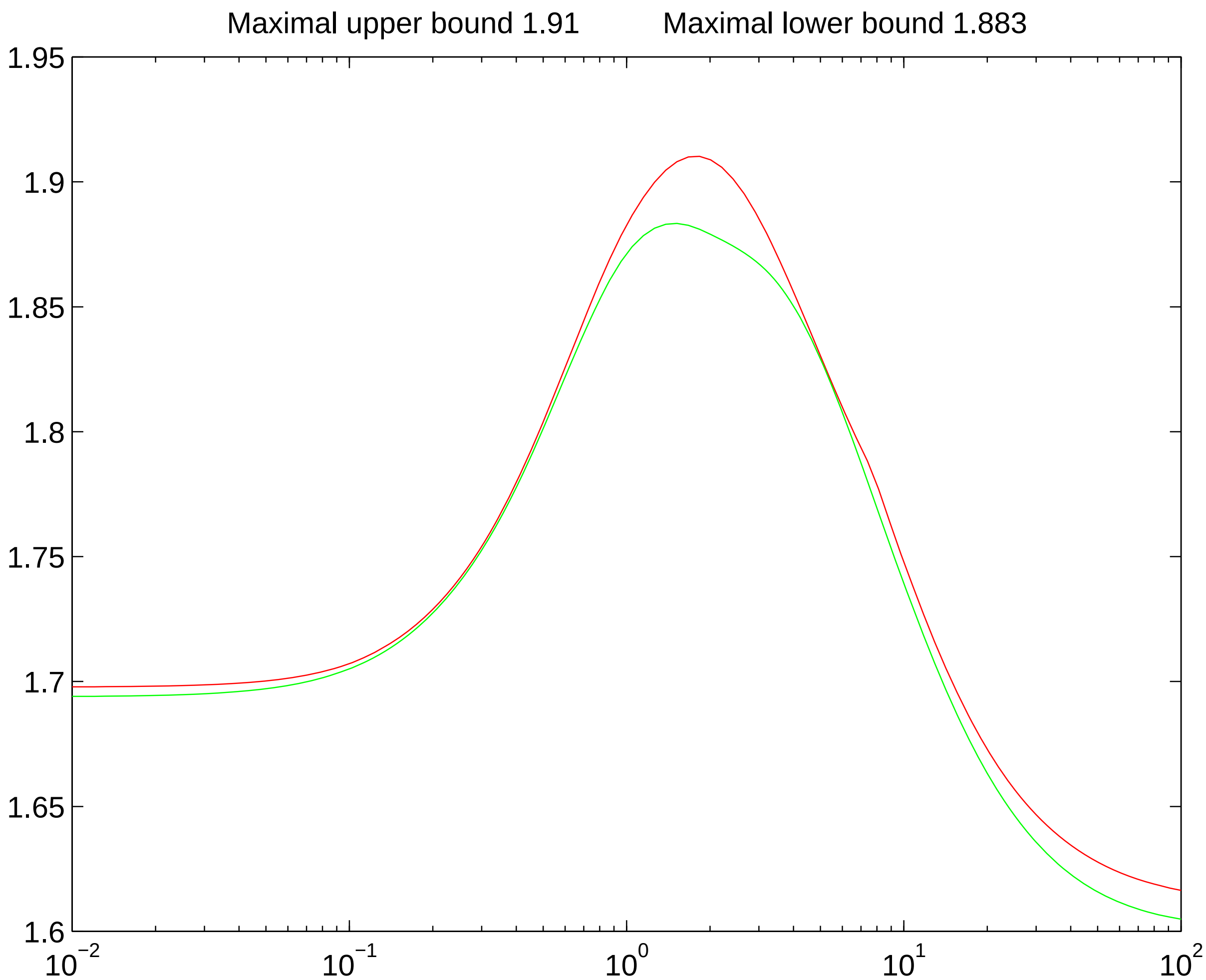


Figure 57: Upper and lower bound of SSV of scaled N.

The first property can be certainly excluded since Figure 55 reveals that $\mu_{\mathbf{\Delta_c}}(M(i\omega)) \leq 2.7$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Let us finally ask ourselves for which size of the uncertainties we can guarantee a robust performance level of 2.

For that purpose let us plot (Figure 57) the SSV of

$$\begin{pmatrix} M & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} 0.5I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0.5M & N_{12} \\ 0.5N_{21} & N_{22} \end{pmatrix}.$$

Since the upper bound is not larger than 1.92 for all frequencies, we conclude

$$\|S(\Delta, N)\|_\infty \leq 1.92 \text{ for all } \Delta \in \frac{0.5}{1.92}\mathbf{\Delta} \approx 0.26\mathbf{\Delta}.$$

Exercises

1) Look at a standard tracking configuration for a system $G(I + \Delta W)$ with input multiplicative uncertainty and a controller K that is described as

$$y = G(I + \Delta W)u, \quad \|\Delta\|_\infty < 1, \quad u = K(r - y).$$

The performance is as desired if the transfer matrix from references r to weighted error $V(r-y)$ has an H_∞-norm smaller than 1. Here G, K, Δ, V, W are LTI system and the latter three are stable.

   a) Set up the generalized plant. Show that the weighted closed-loop transfer matrix has the structure $\begin{pmatrix} -M_1 G & M_1 \\ -M_2 G & M_2 \end{pmatrix}$ by computing M_1 and M_2. Formulate the μ-tests for robust stability, nominal performance and robust performance.

   b) Now let all LTI systems G, K, Δ, V, W be SISO and define $S = (I + KG)^{-1}$, $T = (I + KG)^{-1}KG$. Show that K achieves robust performance iff

$$|V(i\omega)S(i\omega)| + |W(i\omega)T(i\omega)| \leq 1 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}. \tag{6.1}$$

   (Hint: This is a SSV-problem for rank one matrices!) If a controller achieves robust stability and nominal performance, what can you conclude about robust performance? How would you design robustly performing controllers by solving an H_∞ problem?

   c) Let's return to the MIMO case. Suppose that G is square and has a proper inverse G^{-1}. Show that the SSV for the robust performance test is (at frequency ω) bounded from above by

$$\|G(i\omega)\|\|G(i\omega)^{-1}\|\|M_1(i\omega)G(i\omega)\| + \|M_2(i\omega)\|.$$

   If a controller achieves robust stability and nominal performance, what can you now conclude for robust performance? Discuss the role of the plant condition number $\|G(i\omega)\|\|G(i\omega)^{-1}\|$!

   d) For any γ, construct complex matrices M_1, M_2 and G such that $\|M_1 G\| \leq 1$, $\|M_2\| \leq 1$, but $\mu(\begin{pmatrix} -M_1 G & M_1 \\ -M_2 G & M_2 \end{pmatrix}) \geq \gamma$. Here, μ is computed with respect to an uncertainty structure with two full blocks. What does this example show? Hint: Construct the example such that $\|G\|\|G^{-1}\|$ is large.

2) Consider the block diagram in Figure 58 where G and H are described by the transfer functions

$$G(s) = \frac{1}{(0.05s + 1)^2} \quad \text{and} \quad H(s) = \frac{200}{10s + 1}.$$

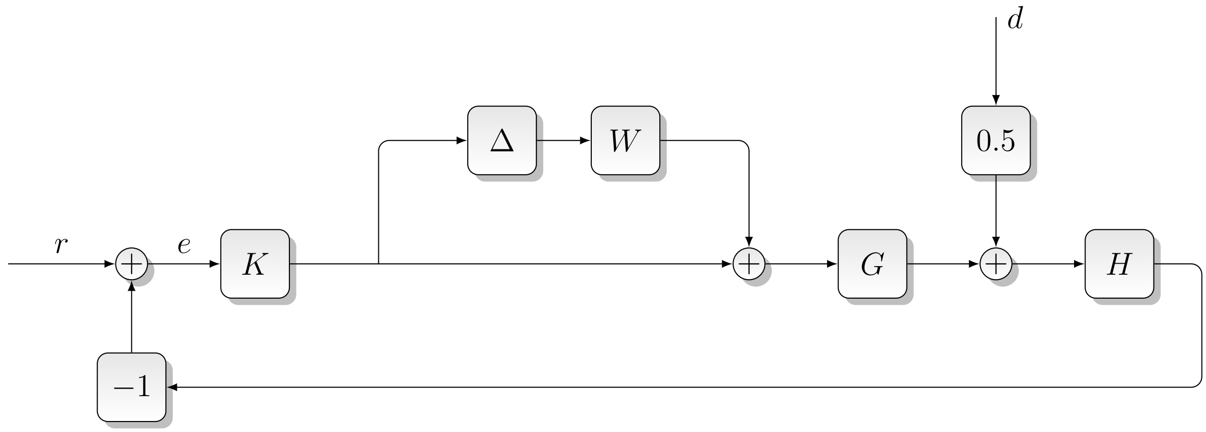


Figure 58: Tracking interconnection for exercise 2).

The magnitude of the uncertainty is not larger than 1% at low frequencies, it does not exceed 100% at 30 rad/sec, and for larger frequencies it increases by 40 db per decade.

a) Design a weighting W that captures the specifications on the uncertainty, and build the open-loop interconnection that corresponds to the block diagram.

b) Consider static gain controllers $u = Ky$ with K in the interval $[1, 3]$. Which controllers stabilize the interconnection? What do you observe if you increase the gain of the controller for the tracking and disturbance suppression behavior of the controlled system? What is the effect on robust stability with respect to the given class of uncertainties?

c) With performance weightings of the form

$$a * \frac{s/b + 1}{s/c + 1}, \quad a, b, c \text{ real},$$

for the channels $d \to e$ and $r \to e$, design a controller with the H_∞ algorithm to achieve the following performance specifications:

i. Disturbance attenuation of a factor 100 up to 0.1 rad/sec.

ii. Zero steady state response of tracking error (smaller than 10^{-4}) for a references step input.

iii. Bandwidth (frequency where magnitude plot first crosses $1/\sqrt{2} \approx -3\,\text{dB}$ from below) from reference input to tracking error between 10 rad/sec and 20 rad/sec.

iv. Overshoot of tracking error is less than 7% in response to step reference.

Provide magnitude plots of all relevant transfer functions and discuss the results.

d) For the design you performed in 2c), what is the maximal size γ_* of uncertainties that do not destabilize the controlled system (stability margin). Compute a destabilizing perturbation of size larger than γ_*.

e) Extend the H_∞ specification of 2c) by the uncertainty channel and perform a new design with the same performance weightings. To what amount do you need to give up the specifications to guarantee robust stability? If you compare the two designs, what do you conclude about performing a nominal design without taking robustness specifications into account?

7 Synthesis of H_∞ Controllers

In this section we provide a self-contained and elementary route to solve the H_∞-problem. We first describe how to bound or compute the H_∞-norm of stable transfer matrix in terms of a suitable Hamiltonian matrix. Then we present a classical result, the so-called Bounded Real Lemma, that characterizes (in terms of a state-space realization) when a given transfer matrix has an H_∞-norm which is strictly smaller than a number γ. On the basis of the Bounded Real Lemma, we will derive the celebrated solution of the H_∞ control problem in terms of two algebraic Riccati equations and a coupling condition on their solutions. We sacrifice generality to render most of the derivations as elementary as possible.

7.1 The Algebraic Riccati Equation and Inequality

The basis for our approach to the H_∞ problem is the algebraic Riccati equation or inequality. It occurs in proving the Bounded Real Lemma and it comes back in getting to the Riccati solution of the H_∞ problem.

Given symmetric matrices $R \geq 0$ and Q, we consider the strict algebraic Riccati inequality

$$A^T X + XA + XRX + Q < 0 \qquad \text{(ARI)}$$

and the corresponding algebraic Riccati equation

$$A^T X + XA + XRX + Q = 0. \qquad \text{(ARE)}$$

Note that X is always assumed to be real symmetric or complex Hermitian. Moreover, we allow for a general indefinite Q.

It will turn out that the solutions of the ARE with the property that $A + RX$ has all its eigenvalues in $\mathbb{C}_<$ or in $\mathbb{C}_>$ play a special role. Such a solutions are called *stabilizing* or *anti-stabilizing*. If (A, R) is controllable, we can summarize the results in this section as follows: The ARE or ARI have solutions if and only if a certain Hamiltonian matrix defined through A, R, Q has no eigenvalues on the imaginary axis. If the ARI or the ARE has a solution, there exists a unique stabilizing solution X_- and a unique anti-stabilizing solution X_+ of the ARE, and all other solutions X of the ARE or ARI satisfy $X_- \leq X \leq X_+$. Here is the main result whose proof is given in the appendix.

Theorem 7.1 *Suppose that Q is symmetric, that R is positive semi-definite, and that (A, R) is controllable. Define the Hamiltonian matrix*

$$H := \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}.$$

Then the following statements are equivalent:

(a) H has no eigenvalues on the imaginary axis.

(b) $A^TX + XA + XRX + Q = 0$ has a (unique) stabilizing solution X_-.

(c) $A^TX + XA + XRX + Q = 0$ has a (unique) anti-stabilizing solution X_+.

(d) $A^TX + XA + XRX + Q < 0$ has a symmetric solution X.

If one and hence all of these conditions are satisfied, then

$$\text{any solution } X \text{ of the ARE or ARI satisfies } \; X_- \le X \le X_+.$$

We conclude that the stabilizing solution is the smallest among all solutions of the ARE and the anti-stabilizing solution is the largest.

Remark. Note that H has a specific structure: The off-diagonal blocks are symmetric, and the second block on the diagonal results from the first by reversing the sign and transposing. Any such matrix is called a *Hamiltonian* matrix.

If (A, R) is only stabilizable, X_+ does, in general, not exist. All other statements, however, remain true. Here is the precise results that is proved in the appendix.

Theorem 7.2 *Suppose that all hypothesis in Theorem 7.1 hold true but that (A, R) is only stabilizable. Then the following statements are equivalent:*

(a) H has no eigenvalues on the imaginary axis.

(b) $A^TX + XA + XRX + Q = 0$ has a (unique) stabilizing solution X_-.

(c) $A^TX + XA + XRX + Q < 0$ has a symmetric solution X.

If one and hence all of these conditions are satisfied, then

$$\text{any solution } X \text{ of the ARE or ARI satisfies } \; X_- \le X.$$

The proof reveals that it is not difficult to construct a solution once one has verified that H has no eigenvalues on the imaginary axis. We sketch the typical algorithm that is used in software packages like Matlab.

Indeed, let H have no eigenvalues in $\mathbb{C}_=$. Then it has n eigenvalues in $\mathbb{C}_<$ and n eigenvalues in $\mathbb{C}_>$ respectively. We can perform a Schur decomposition to obtain a unitary matrix T with

$$T^*HT = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where M_{11} of size $n \times n$ is stable and M_{22} of size $n \times n$ is anti-stable. Partition T into four $n \times n$ blocks as

$$T = \begin{pmatrix} U & * \\ V & * \end{pmatrix}.$$

The proof of Theorem 7.2 reveals that U is non-singular, and that the stabilizing solution of the ARE is given by

$$X = VU^{-1}.$$

If the Schur decomposition is chosen such that M_{11} has all its eigenvalues in $\mathbb{C}_>$ and M_{22} is stable, then the same procedure leads to the anti-stabilizing solution.

If Q is negative semi-definite, the eigenvalues of the Hamiltonian matrix on the imaginary axis are just given by uncontrollable or unobservable modes. The exact statement reads as follows.

Lemma 7.3 *If $R \geq 0$ and $Q \leq 0$ then*

$$\lambda \in \mathbb{C}_= \ \ \textit{is an eigenvalue of} \ \ H = \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}$$

if and only if

$$\lambda \in \mathbb{C}_= \ \textit{is an uncontrollable mode of} \ (A, R) \ \textit{or an unobservable mode of} \ (A, Q).$$

Proof. $i\omega$ is an eigenvalue of H if and only if

$$H - i\omega I = \begin{pmatrix} A - i\omega I & R \\ -Q & -A^T - i\omega I \end{pmatrix} = \begin{pmatrix} A - i\omega I & R \\ -Q & -(A - i\omega I)^* \end{pmatrix}$$

is singular.

If $i\omega$ is an uncontrollable mode of (A, R), then $\begin{pmatrix} A - i\omega I & R \end{pmatrix}$ does not have full row rank; if it is an unobservable mode of (A, Q), then $\begin{pmatrix} A - i\omega I \\ -Q \end{pmatrix}$ does not have full column rank; in both cases we infer that $H - i\omega I$ is singular.

Conversely, suppose that $H - i\omega I$ is singular. Then there exist x and y, not both zero, with

$$\begin{pmatrix} A - i\omega I & R \\ -Q & -(A - i\omega)^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

This implies

$$(A - i\omega I)x + Ry = 0 \;\text{ and }\; -Qx - (A - i\omega I)^* y = 0. \tag{7.1}$$

Left-multiply the first equation with y^* and the second equation with x^* to get

$$\begin{aligned} y^*(A - i\omega I)x + y^*Ry &= 0 \\ -x^*Qx - y^*(A - i\omega I)x = -x^*Qx - x^*(A - i\omega I)^* y &= 0. \end{aligned}$$

This leads to

$$y^*Ry = x^*Qx.$$

Since $R \geq 0$ and $Q \leq 0$, we infer $Qx = 0$ and $Ry = 0$. Then (7.1) implies $(A - i\omega I)x = 0$ and $(A - i\omega I)^* y = 0$. If $x \neq 0$, $i\omega$ is an unobservable mode of (A, Q), if $y \neq 0$, it is an uncontrollable mode of (A, R).

■

Hence, if (A, R) is stabilizable, if $Q \leq 0$, and if (A, Q) does not have unobservable modes on the imaginary axis, the corresponding Hamiltonian matrix does not have eigenvalues on the imaginary axis such that the underlying ARE has a stabilizing solution and the ARI is solvable as well.

7.2 Computation of H_∞ Norms

Consider the strictly proper transfer matrix M with realization

$$M(s) = C(sI - A)^{-1}B$$

where A is stable. Recall that the H_∞-norm of M is defined by

$$\|M\|_\infty := \sup_{\omega \in \mathbb{R}} \|M(i\omega)\|.$$

In general it is not advisable to compute the H_∞-norm of M by solving this optimization problem. In this section we clarify how one can arrive at a more efficient computation of this norm by looking, instead, at the following problem: Characterize in terms of A, B, C whether the inequality

$$\|M\|_\infty < 1 \tag{7.2}$$

is true or not. Just by the definition of the H_∞ norm, (7.2) is equivalent to

$$\|M(i\omega)\| < 1 \;\text{ for all }\; \omega \in \mathbb{R} \cup \{\infty\}.$$

Since M is strictly proper, this inequality is always true for $\omega = \infty$. Hence it remains to consider

$$\|M(i\omega)\| < 1 \;\text{ for all }\; \omega \in \mathbb{R}. \tag{7.3}$$

If follows by continuity that this is true if and only if

$$\det(M(i\omega)^* M(i\omega) - I) \neq 0 \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{7.4}$$

Indeed, $\|M(i\omega)\| < 1$ implies that the largest eigenvalue of $M(i\omega)^* M(i\omega)$ is smaller than 1 such that $\det(M(i\omega)^* M(i\omega) - I) \neq 0$. Hence (7.3) implies (7.4). Conversely, suppose (7.3) is not true. Then there exists a $\omega_0 \in \mathbb{R}$ for which $\|M(i\omega_0)\| \geq 1$. Consider the real-valued function $\omega \to \|M(i\omega)\|$ which is continuous (as the norm of a rational function without pole). Due to $\lim_{\omega\to\infty} \|M(i\omega)\| = 0$, there exists an $\omega_1 > \omega_0$ with $\|M(i\omega_1)\| < 1$. By the intermediate value theorem, there exists some point in the interval $\omega_* \in [\omega_0, \omega_1]$ with $\|M(i\omega_*)\| = 1$. This implies $\det(M(i\omega_*)^* M(i\omega_*) - I) = 0$ such that (7.4) is not true.

Since M is real rational we have $M(i\omega)^* = M(-i\omega)^T$. If we hence define

$$G(s) := M^T(-s)M(s) - I,$$

(7.4) is the same as

$$\det(G(i\omega)) \neq 0 \quad \text{for all} \quad \omega \in \mathbb{R}.$$

Since

$$M(-s)^T = [C(-sI - A)^{-1}B]^T = B^T(-(sI + A^T)^{-1})C^T = B^T(sI - (-A^T))^{-1}(-C^T),$$

one easily obtains a state-space realization of G as

$$G = \left[\begin{array}{cc|c} A & 0 & B \\ -C^T C & -A^T & 0 \\ \hline 0 & B^T & -I \end{array}\right].$$

Let us now apply the Schur formula[3] to this realization for $s = i\omega$. If we introduce the abbreviation

$$H := \begin{pmatrix} A & 0 \\ -C^T C & -A^T \end{pmatrix} - \begin{pmatrix} B \\ 0 \end{pmatrix} (-I)^{-1} \begin{pmatrix} 0 & B^T \end{pmatrix} = \begin{pmatrix} A & BB^T \\ -C^T C & -A^T \end{pmatrix}, \tag{7.5}$$

we arrive at

$$\det(G(i\omega)) = \frac{\det(-I)}{\det(i\omega I - A)\det(i\omega I + A^T)} \det(i\omega I - H).$$

Now recall that A is stable such that both $\det(i\omega I - A)$ and $\det(i\omega I + A^T)$ do not vanish. Hence

$$\det(G(i\omega)) = 0 \quad \text{if and only if} \quad i\omega \quad \text{is an eigenvalue of} \quad H.$$

Hence (7.4) is equivalent to the fact that H does not have eigenvalues on the imaginary axis. This leads to the following characterization of the H_∞-norm bound $\|M\|_\infty < 1$.

[3]If D is nonsingular, $\det(C(sI - A)^{-1}B + D) = \frac{\det(D)}{\det(sI-A)} \det(sI - (A - BD^{-1}C))$

Theorem 7.4 $\|M\|_\infty < 1$ *if and only if* $\begin{pmatrix} A & BB^T \\ -C^TC & -A^T \end{pmatrix}$ *has no eigenvalues on the imaginary axis.*

To compute $\|M\|_\infty$, we actually need to verify whether, for any given positive number γ,

$$\|C(sI - A)^{-1}B\|_\infty < \gamma \tag{7.6}$$

is valid or not. Indeed, the inequality is the same as

$$\| \left[\frac{1}{\gamma}C\right] (sI - A)^{-1}B\|_\infty < 1 \quad \text{or} \quad \|C(sI - A)^{-1} \left[\frac{1}{\gamma}B\right] \|_\infty < 1 \tag{7.7}$$

such that it suffices to re-scale either B or C by the factor $\frac{1}{\gamma}$ to reduce the test to the one with bound 1. We conclude: (7.6) holds if and only if

$$\begin{pmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{pmatrix} \quad \text{has no eigenvalues on the imaginary axis}$$

or, equivalently,

$$\begin{pmatrix} A & BB^T \\ -\frac{1}{\gamma^2}C^TC & -A^T \end{pmatrix} \quad \text{has no eigenvalues on the imaginary axis.}$$

Why does this result help? It allows to check $\|M\|_\infty < \gamma$, a test which involves computing the norm at infinitely many frequencies, by just verifying whether a Hamiltonian matrix that is defined through the data matrices A, B, C and the bound γ has an eigenvalue on the imaginary axis or not. This allows to compute $\|M\|_\infty$ by bisection (Appendix A).

7.3 The Bounded Real Lemma

The characterization of $\|M\|_\infty < 1$ in terms of the Hamiltonian matrix H is suitable for computing the H_∞-norm of M, but it is not convenient to derive a solution of the H_∞ problem. For that purpose we aim at providing an alternative characterization of $\|M\|_\infty < 1$ in terms of the solvability of Riccati equations or inequalities. In view of our preparations showing a relation of the solvability of the ARE or ARI with Hamiltonians, this is not too surprising.

Theorem 7.5 *Let* $M(s) = C(sI - A)^{-1}B$ *with* A *being stable. Then* $\|M\|_\infty < 1$ *holds if and only if the ARI*

$$A^TX + XA + XBB^TX + C^TC < 0 \tag{7.8}$$

has a solution. This is equivalent to the fact that the ARE

$$A^T X + XA + XBB^T X + C^T C = 0 \tag{7.9}$$

has a stabilizing solution.

We only need to observe that the stability of A implies that (A, BB^T) is stabilizable. Then we can just combine Theorem 7.2 with Theorem 7.4 to obtain Theorem 7.5.

Since the H_∞-norms of $C(sI - A)^{-1}B$ and of $B^T(sI - A^T)^{-1}C^T$ coincide, we can dualize this result.

Theorem 7.6 *Let $M(s) = C(sI - A)^{-1}B$ with A being stable. Then $\|M\|_\infty < 1$ holds if and only if the ARI*

$$AY + YA^T + BB^T + YC^T CY < 0 \tag{7.10}$$

has a solution. This is equivalent to the fact that the ARE

$$AY + YA^T + BB^T + YC^T CY = 0 \tag{7.11}$$

has a stabilizing solution.

Task. Provide the arguments why these statements are true.

Remarks.

- Recall how we reduced the bound (7.6) for some $\gamma > 0$ to (7.7). Hence (7.6) can be characterized by performing the substitutions

$$BB^T \to \frac{1}{\gamma^2} BB^T \quad \text{or} \quad C^T C \to \frac{1}{\gamma^2} C^T C$$

  in all the four AREs or ARIs.

- If X satisfies (7.8), it must be non-singular: Suppose $Xx = 0$ with $x \neq 0$. Then x^T(7.8)$x = x^T BB^T x < 0$. This implies $\|B^T x\| < 0$, a contradiction. If we note that X^{-1}(7.9)X^{-1} implies

$$AX^{-1} + X^{-1}A^T + BB^T + X^{-1}C^T CX^{-1} < 0,$$

  we infer that $Y = X^{-1}$ satisfies (7.10). Conversely, if Y solves (7.10), then Y^{-1} exists and satisfies (7.8).

7.4 The H_∞-Control Problem

Let us consider the generalized plant

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \begin{pmatrix} w \\ u \end{pmatrix}.$$

Recall that y is the measured output available for control, u is the control input, z is the controlled output, and w the disturbance input. We assume that P admits a stabilizing controller such that

$$(A, B_2) \text{ is stabilizable and } (A, C_2) \text{ is detectable.}$$

As controllers we allow for any LTI system

$$u = Ky = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] y$$

specified in the state-space through the parameter matrices A_K, B_K, C_K, D_K.

The goal in H_∞-control is to minimize the H_∞-norm of the transfer function $w \to z$ by using stabilizing controllers. With the previous notation for the controlled closed-loop system

$$z = S(P, K)w = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] w,$$

the intention is to minimize

$$\|S(P, K)\|_\infty = \left\| \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] \right\|_\infty$$

over all K that stabilizes P, i.e., which render $\mathcal{A}$ stable.

Similarly as for just determining the H_∞-norm of a transfer matrix, we rather consider the so-called **sub-optimal H_∞ control problem**: Given the number $\gamma > 0$, find a controller K such that

$$K \text{ stabilizes } P \text{ and achieves } \|S(P, K)\|_\infty < \gamma$$

or conclude that no such controller exists.

As usual, we can rescale to the bound 1 by introducing weightings. This amounts to substituting P by either one of

$$\begin{pmatrix} \frac{1}{\gamma}P_{11} & \frac{1}{\gamma}P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\gamma}P_{11} & P_{12} \\ \frac{1}{\gamma}P_{21} & P_{22} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\gamma}P_{11} & \frac{1}{\sqrt{\gamma}}P_{12} \\ \frac{1}{\sqrt{\gamma}}P_{21} & P_{22} \end{pmatrix}$$

that read in the state-space as

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline \frac{1}{\gamma}C_1 & \frac{1}{\gamma}D_{11} & \frac{1}{\gamma}D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad \left[\begin{array}{c|cc} A & \frac{1}{\gamma}B_1 & B_2 \\ \hline C_1 & \frac{1}{\gamma}D_{11} & D_{12} \\ C_2 & \frac{1}{\gamma}D_{21} & D_{22} \end{array}\right], \quad \left[\begin{array}{c|cc} A & \frac{1}{\sqrt{\gamma}}B_1 & B_2 \\ \hline \frac{1}{\sqrt{\gamma}}C_1 & \frac{1}{\gamma}D_{11} & \frac{1}{\sqrt{\gamma}}D_{12} \\ C_2 & \frac{1}{\sqrt{\gamma}}D_{21} & D_{22} \end{array}\right].$$

Hence the problem is re-formulated as follows: Try to find a controller K such that

$$K \text{ stabilizes } P \text{ and achieves } \|S(P,K)\|_\infty < 1. \tag{7.12}$$

The conditions (7.12) read in the state-space as

$$\lambda(\mathcal{A}) \subset \mathbb{C}_< \text{ and } \|\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D}\|_\infty < 1. \tag{7.13}$$

Note that a K might or might not exist. Hence our goal is to provide verifiable conditions formulated in terms of the generalized plant P for the existence of such a controller K. If a controller is known to exist, we also need to devise an algorithm that allows to construct a suitable controller K which renders the conditions in (7.12) or (7.13) satisfied.

7.5 H_∞-Control for a Simplified Generalized Plant Description

The generalized plant reads as

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u. \end{aligned}$$

The derivation of a solution of the H_∞-problem in its generality is most easily obtained with LMI techniques. In these notes we only consider the so-called regular problem. This amounts to the hypothesis that

$$D_{12} \text{ has full column rank and } D_{21} \text{ has full row rank.}$$

These assumptions basically imply that the full control signal u appears via $D_{12}u$ in z, and that the whole measured output signal y is corrupted via $D_{21}w$ by noise.

In order to simplify both the derivation and the formulas, we confine the discussion to a generalized plant with the following stronger properties:

$$D_{11} = 0, \quad D_{22} = 0, \quad D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^T = \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{7.14}$$

Hence we assume that both P_{11} and P_{22} are strictly proper. Moreover, D_{12} does not only have full column rank, but its columns are orthonormal, and they are orthogonal to the columns of C_1. Similarly, the rows of D_{21} are orthonormal, and they are orthogonal to the rows of B_1.

7.6 The State-Feedback Problem

Let us first concentrate on the specific control structure

$$u = Fx$$

what is often denoted as *static state-feedback*. The closed-loop system then reads as

$$\begin{aligned} \dot{x} &= (A + B_2F)x + B_1w \\ z &= (C_1 + D_{12}F)x. \end{aligned}$$

Hence the static-state feedback H_∞ control problem is formulated as follows: Find an F which renders the following two conditions satisfied:

$$\lambda(A + B_2F) \subset \mathbb{C}_< \quad \text{and} \quad \|(C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_1\|_\infty < 1. \tag{7.15}$$

7.6.1 Solution in Terms of Riccati Inequalities

The gain F satisfies both conditions if and only if there exists a Y with

$$Y > 0, \;\; (A+B_2F)Y + Y(A+B_2F)^T + B_1B_1^T + Y(C_1+D_{12}F)^T(C_1+D_{12}F)Y < 0. \tag{7.16}$$

Indeed, if F satisfies (7.15), we can apply Theorem 7.6 to infer that the ARI in (7.16) has a symmetric solution Y. Since the inequality implies $(A + B_2F)Y + Y(A + B_2F)^T < 0$, and since $A + B_2F$ is stable, we infer that Y is actually positive definite. Conversely, if Y satisfies (7.16), then $(A + B_2F)Y + Y(A + B_2F)^T < 0$ implies that $A + B_2F$ is stable. Again by Theorem 7.6 we arrive at (7.15).

In a next step, we eliminate F from (7.16). This will be possible on the basis of the following lemma.

Lemma 7.7 *For any F,*

$$\begin{aligned} (A + B_2F)Y + Y(A + B_2F)^T + B_1B_1^T + Y(C_1 + D_{12}F)^T(C_1 + D_{12}F)Y \geq \\ \geq AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y. \end{aligned}$$

Equality holds if and only if

$$FY + B_2 = 0.$$

Proof. Note that (7.14) implies

$$Y(C_1 + D_{12}F)^T(C_1 + D_{12}F)Y = YC_1^TC_1Y + YF^TFY.$$

Moreover, we exploit (completion of the squares)

$$(B_2FY) + (B_2FY)^T + YF^TFY = -B_2B_2^T + (FY + B_2^T)^T(FY + B_2^T).$$

Both equations imply

$$(A + B_2F)Y + Y(A + B_2F)^T + B_1B_1^T + Y(C_1 + D_{12}F)^T(C_1 + D_{12}F)Y =$$
$$= AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y + (FY + B_2^T)^T(FY + B_2^T).$$

Since $(FY + B_2^T)^T(FY + B_2^T) \geq 0$ and since $(FY + B_2^T)^T(FY + B_2^T) = 0$ if and only if $FY + B_2^T = 0$, the proof is finished.

■

Due to this lemma, any Y that satisfies (7.16) also satisfies

$$Y > 0, \quad AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y < 0. \tag{7.17}$$

Note that the resulting ARI is independent of F! Conversely, if Y is a matrix with (7.17), it is non-singular such that we can set

$$F := -B_2^TY^{-1}.$$

Since $FY + B_2$ vanishes, we can again apply Lemma 7.7 to see that (7.17) implies (7.16). To conclude, there exists an F and a Y with (7.16) if and only if there exists a Y with (7.17).

Let us summarize what we have found in the following Riccati inequality solution of the state-feedback H_∞ problem.

Theorem 7.8 *The gain F solves the state-feedback H_∞-problem (7.15) if and only if there exists a positive definite solution Y of the ARI*

$$AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y < 0. \tag{7.18}$$

If $Y > 0$ is any solution of this ARI, then the gain

$$F := -B_2^TY^{-1}$$

renders $A + B_2F$ stable and leads to $\|(C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_1\|_\infty < 1$.

At this point it is unclear how we can test whether (7.18) has a positive definite solution. One possibility is as follows: Observe that $Y > 0$ and (7.18) are equivalent to (Schur complement)

$$Y > 0, \quad \begin{pmatrix} AY + YA^T + B_1B_1^T - B_2B_2^T & YC_1^T \\ C_1Y & -I \end{pmatrix} < 0.$$

These are two linear matrix inequalities and, hence, they can be readily solved by existing software.

7.6.2 Solution in Terms of Riccati Equations

There is an alternative. We can reduce the solvability of (7.18) to a test for a certain Hamiltonian matrix via Theorem 7.1. For that purpose we need an additional technical hypothesis; we have to require that $(A^T, C_1^T C_1)$ is controllable. By the Hautus test, this is the same as (A, C_1) being observable. (Why?) It is important to note that this assumption is purely technical and makes it possible to provide a solution of the state-feedback H_∞ problem by Riccati equations; it might happen that this property fails to hold such that one has to rely on alternative techniques.

Theorem 7.9 *Suppose that*

$$(A, C_1) \quad \textit{is observable.} \tag{7.19}$$

Then there exists an F that solves the state-feedback H_∞-problem (7.15) if and only if the Riccati equation

$$AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^TC_1Y = 0 \tag{7.20}$$

has an anti-stabilizing solution Y_+, and this anti-stabilizing solution is positive definite. If $Y_+ > 0$ denotes the anti-stabilizing solution of the ARE, the gain

$$F := -B_2^T Y_+^{-1}$$

leads to (7.15).

Proof. If there exists an F that satisfies (7.15), the ARI (7.18) has a solution $Y > 0$. By Theorem 7.2, the ARE (7.20) has an anti-stabilizing solution Y_+ that satisfies $Y \leq Y_+$. Hence Y_+ exists and is positive definite.

Conversely, suppose Y_+ exists and is positive definite. With $F = -B_2^T Y_+^{-1}$ we infer $FY_+ + B_2 = 0$ and hence

$$AY_+ + Y_+A^T + GG^T - B_2B_2^T + Y_+C_1^TC_1Y_+ + (FY_+ + B_2^T)^T(FY_+ + B_2^T) = 0.$$

By Lemma 7.7, we arrive at

$$(A + B_2F)Y_+ + Y_+(A + B_2F)^T + B_1B_1^T + Y_+(C_1 + D_{12}F)^T(C_1 + D_{12}F)Y_+ = 0. \tag{7.21}$$

Let us first show that $A + B_2F$ is stable. For that purpose let $(A + B_2F)^Tx = \lambda x$ with $x \neq 0$. Now look at $x^*(7.21)x$:

$$\text{Re}(\lambda)x^*Y_+x + x^*B_1B_1^Tx + x^*Y_+(C_1 + D_{12}F)^T(C_1 + D_{12}F)Y_+x = 0.$$

Since $Y_+ \geq 0$, this implies $\text{Re}(\lambda) \geq 0$. Let us now exclude $\text{Re}(\lambda) = 0$. In fact, this condition implies $(C_1 + D_{12}F)Y_+x = 0$. If we left-multiply D_{12}, we can exploit (7.14) to

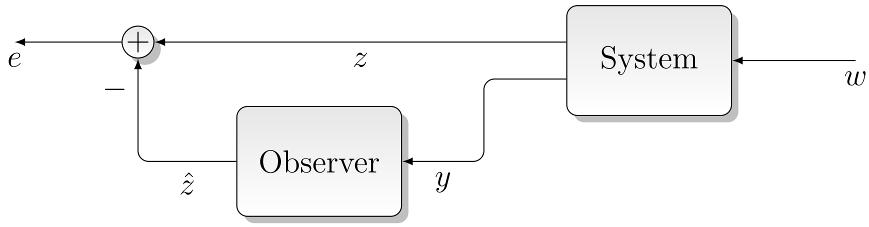


Figure 59: H_∞-observer design.

infer $FY_+x = 0$. This implies $B_2^Tx = 0$ and hence $A^Tx = \lambda x$. Since (A, B_2) is stabilizable, we obtain $\mathrm{Re}(\lambda) < 0$, a contradiction. Therefore, the real part of λ must be negative, what implies that $A + B_2F$ is stable.

Since
$$A + B_2F + Y_+(C_1 + D_{12}F)^T(C_1 + D_{12}F) = A + Y_+C_1^TC_1,$$
we infer that Y_+ is the anti-stabilizing solution of (7.21). Hence the corresponding Hamiltonian matrix has no eigenvalues on the imaginary axis (Theorem 7.1) such that (7.15) follows (Theorem 7.4).

■

7.7 H_∞-Observer Design

Suppose a given system is affected by a genralized disturbance w. Based on the measurement output y, the goal is to estimate the output z of the system as closely as possible.

This should be achieved by an observer, a copy of the system (while neglecting the unknown disturbance input) that processes the measured output y to generate the estimate $\hat{z}$, which leads to the estimation error
$$e = z - \hat{z}.$$

In H_∞-observer design, the quality of the observer is measured in terms of the H_∞-norm of the transfer matrix defined by $w \to e$.

Let the system be described by
$$\dot{x} = Ax + B_1w,\ z = C_1x,\ y = C_2x + D_{21}w.$$

An observer is then defined with suitable gain L as
$$\dot{\hat{x}} = A\hat{x} + L(C_2\hat{x} - y),\ \hat{z} = C_1\hat{x}.$$

For the error dynamics (with state $\xi := x - \hat{x}$) as described by
$$\dot{\xi} = (A + LC_2)\xi + (B_1 + LD_{21})w,\ e = C_1\xi,$$

the goal is to achieve with some $\gamma > 0$:

$$\mathrm{eig}(A + LC_2) \subset \mathbb{C}^- \text{ and } \|C_1(sI - A - LC_2)^{-1}(B_1 + LD_{21})\|_\infty < \gamma \tag{OD}$$

Any observer that guarantees (OD) leads to the following properties:

- For nonzero initial condition and the absence of the disturbances, the observer state converges to the system state asymptotically:

$$\lim_{t\to\infty}(x(t) - \hat{x}(t)) = 0$$

- For zero initial conditions, the estimation error $z - \hat{z}$ is endered small in worst-case for all disturbances w, in the sense that the H_∞-norm of $w \to (z - \hat{z})$ is smaller than γ.
- The smallest achievable γ is called optimal H_∞-estimation level.
- Recall all our earlier discussions concerning the interpretation of this measure! Morevoer, dynamic filters for capturing spectral properties of the disturbance are again assumed to be absorbed into the system description at the input.

The following result is easily established by duality.

Theorem 7.10 *Let (A, C_2) be detectable, $B_1 D_{21}^T = 0$, $D_{21}D_{21}^T = I$ and suppose that (A, B_1) is controllable. Then there exists an observer gain L which achieves (OD) if and only if the anti-stabilizing solution X_+ of the ARE*

$$A^T X + XA + XB_1B_1^T X - \gamma^{-2}C_1^T C_1 - C_2^T C_2 = 0$$

exists and is positive definite. If the anti-stabilizing solution X_+ of the ARE exists and is positife definite, $L := -X_+^{-1}C_2^T$ satisfies (OD).

For the proof just observe that (OD) is equivalent to

$$\mathrm{eig}(A^T + C_2^T L^T) \subset \mathbb{C}^- \text{ and } \|(B_1^T + D_{21}^T L^T)(sI - A^T - C_2^T L^T)^{-1}C_1^T\|_\infty < \gamma.$$

Then apply Theorem 7.9 for $(A^T, C_1^T, C_2^T, B_1^T, D_{21}^T)$ substituting $(A, B_1, B_2, C_1, D_{12})$.

7.8 H_∞-Control by Output-Feedback

Let us now come back to the H_∞-problem by output feedback control. This amounts to finding the matrices A_K, B_K, C_K, D_K such that the conditions (7.13) are satisfied.

We proceed as in the state-feedback problem. We use the Bounded Real Lemma to rewrite the H_∞ norm bound into the solvability of a Riccati inequality, and we try to eliminate the controller parameters to arrive at verifiable conditions.

7.8.1 Solution in Terms of Riccati Inequalities

For the derivation to follow we assume that the controller is strictly proper: $D_K = 0$. The proof for a controller with $D_K \neq 0$ is only slightly more complicated and fits much better into the LMI framework what will be discussed in the LMI course.

There exists a controller A_K, B_K, C_K, $D_K = 0$ that satisfies (7.13) if and only if there exists some $\mathcal{X}$ with

$$\mathcal{X} > 0, \quad \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X} + \mathcal{C}^T\mathcal{C} < 0. \tag{7.22}$$

Indeed, (7.13) implies the existence of a symmetric solution of the ARI in (7.22) by Theorem 7.5. Since $\mathcal{A}$ is stable, $\mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} < 0$ implies that $\mathcal{X} > 0$. Conversely, (7.22) implies $\mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} < 0$ for $\mathcal{X} > 0$. Hence $\mathcal{A}$ is stable and we arrive, again by Theorem 7.5, at (7.13).

Let us now suppose that (7.22) is valid. Partition $\mathcal{X}$ and $\mathcal{X}^{-1}$ in the same fashion as $\mathcal{A}$ to obtain

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & \widehat{X} \end{pmatrix} \quad \text{and} \quad \mathcal{X}^{-1} = \begin{pmatrix} Y & V \\ V^T & \widehat{Y} \end{pmatrix}.$$

It is then obvious that

$$\mathcal{R} = \begin{pmatrix} X & U \\ I & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{S} = \begin{pmatrix} I & 0 \\ Y & V \end{pmatrix} \quad \text{satisfy} \quad \mathcal{S}\mathcal{X} = \mathcal{R}.$$

We can assume without loss of generality that the size of A_K is not smaller than n. Hence the right upper block U of $\mathcal{X}$ has more columns than rows. This allows to assume that U is of full row rank; if not true one just needs to slightly perturb this block without violating the *strict* inequalities (7.22). Then we conclude that $\mathcal{R}$ has full row rank. Due to $\mathcal{S}\mathcal{X} = \mathcal{R}$, also $\mathcal{S}$ has full row rank.

Let us now left-multiply both inequalities in (7.22) with $\mathcal{S}$ and right-multiply with $\mathcal{S}^T$. Since $\mathcal{S}$ has full row rank and if we exploit $\mathcal{S}\mathcal{X} = \mathcal{R}$ or $\mathcal{X}\mathcal{S}^T = \mathcal{R}^T$, we arrive at

$$\mathcal{S}\mathcal{R}^T > 0, \quad \mathcal{S}\mathcal{A}^T\mathcal{R}^T + \mathcal{R}\mathcal{A}\mathcal{S}^T + \mathcal{R}\mathcal{B}\mathcal{B}^T\mathcal{R}^T + \mathcal{S}\mathcal{C}^T\mathcal{C}\mathcal{S}^T < 0. \tag{7.23}$$

The appearing blocks are most easily computed as follows:

$$
\begin{aligned}
\mathcal{S}\mathcal{R}^T &= \begin{pmatrix} X & I \\ YX + VU^T & Y \end{pmatrix} \\
\mathcal{R}\mathcal{A}\mathcal{S}^T &= \begin{pmatrix} XA + UB_K C_2 & XAY + UB_K C_2 Y + XB_2 C_K V^T + UA_K V^T \\ A & AY + B_2 C_K V^T \end{pmatrix} \\
\mathcal{R}\mathcal{B} &= \begin{pmatrix} XB_1 + UB_K D_{21} \\ B_1 \end{pmatrix} \\
\mathcal{C}\mathcal{S}^T &= \begin{pmatrix} C_1 & C_1 Y + D_{12} C_K V^T \end{pmatrix}.
\end{aligned}
$$

Let us now recall a relation between X, Y, U, V and define the new variables F and L as in

$$
YX + VU^T = I, \quad L = X^{-1} U B_K \ \text{ and } \ F = C_K V^T Y^{-1}. \tag{7.24}
$$

Then the formulas simplify to

$$
\begin{aligned}
\mathcal{S}\mathcal{R}^T &= \begin{pmatrix} X & I \\ I & Y \end{pmatrix} \\
\mathcal{R}\mathcal{A}\mathcal{S}^T &= \begin{pmatrix} X(A + LC_2) & X(A + LC_2 + B_2 F)Y + UA_K V^T \\ A & (A + B_2 F)Y \end{pmatrix} \\
\mathcal{R}\mathcal{B} &= \begin{pmatrix} X(B_1 + LD_{21}) \\ B_1 \end{pmatrix} \\
\mathcal{C}\mathcal{S}^T &= \begin{pmatrix} C_1 & (C_1 + D_{12} F)Y \end{pmatrix}.
\end{aligned}
$$

We conclude that (7.23) read as

$$
\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0
$$

and

$$
\begin{aligned}
&\begin{pmatrix} (A + LC_2)^T X + X(A + LC_2) & A^T + X(A + LC_2 + B_2 F)Y + UA_K V^T \\ A + [X(A + LC_2 + B_2 F)Y + UA_K V^T]^T & (A + B_2 F)Y + Y(A + B_2 F)^T \end{pmatrix} + \\
&\qquad + \begin{pmatrix} X(B_1 + LD_{21})(B_1 + LD_{21})^T X & X(B_1 + LD_{21})B_1^T \\ B_1 (B_1 + LD_{21})^T X & B_1 B_1^T \end{pmatrix} + \\
&\qquad\quad + \begin{pmatrix} C_1^T C_1 & C_1^T (C_1 + D_{12} F)Y \\ Y(C_1 + D_{12} F)^T C_1 & Y(C_1 + D_{12} F)^T (C_1 + D_{12} F)Y \end{pmatrix} < 0.
\end{aligned} \tag{7.25}
$$

If we pick out the diagonal blocks of the last inequality, we conclude

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0 \tag{7.26}$$

$$(A + LC_2)^T X + X(A + LC_2) + X(B_1 + LD_{21})(B_1 + LD_{21})^T X + C_1^T C_1 < 0 \tag{7.27}$$

$$(A + B_2F)Y + Y(A + B_2F)^T + B_1B_1^T + Y(C_1 + D_{12}F)^T(C_1 + D_{12}F)Y < 0. \tag{7.28}$$

We can apply Lemma 7.7 to (7.28) and a dual version to (7.27). This implies that X and Y satisfy (7.26) and the two algebraic Riccati inequalities

$$A^T X + XA + XB_1B_1^T X + C_1^T C_1 - C_2^T C_2 < 0 \tag{7.29}$$

$$AY + YA^T + B_1B_1^T - B_2B_2^T + YC_1^T C_1 Y < 0. \tag{7.30}$$

We have shown: If there exists a controller that renders (7.13) satisfies, then there exist symmetric matrices X and Y that satisfy the two algebraic Riccati inequalities (7.29)-(7.30), and that these two matrices are coupled as (7.26). Note that these conditions are, again, formulated only in terms of the matrices defining the generalized plant; the controller parameters have been eliminated. It will turn out that one can reverse the arguments: If X and Y satisfy (7.26),(7.29)-(7.30), one can construct a controller that leads to (7.13).

Suppose X and Y satisfy (7.26),(7.29)-(7.30). Due to (7.26), X and Y are positive definite and hence nonsingular. If we apply Lemma 7.7 to (7.30) and a dual version to (7.29), we conclude that

$$L := -X^{-1}C_2^T \quad \text{and} \quad F := -B_2^T Y^{-1}$$

lead to (7.26)-(7.28). Again due to (7.26), $I - YX$ is non-singular as well. Hence we can find non-singular square matrices U and V satisfying $VU^T = I - YX$; take for example $U = I$ and $V = I - YX$. If we set

$$B_K = U^{-1}XL \quad \text{and} \quad C_K = FYV^{-T}, \tag{7.31}$$

we infer that the relations (7.24) are valid. If we now consider (7.25), we observe that the only undefined block on the left-hand side is A_K and this appears as UA_KV^T in the off-diagonal position. We also observe that the diagonal blocks of this matrix are negative definite. If we choose A_K to render the off-diagonal block zero, we infer that (7.25) is valid. This is clearly achieved for

$$A_K = -U^{-1}[A^T + X(A + LC_2 + B_2F)Y + X(B_1 + LD_{21})B_1^T + C_1^T(C_1 + D_{12}F)Y]V^{-T}. \tag{7.32}$$

We arrive back to (7.23). Now $\mathcal{S}$ is square and non-singular. We can hence define $\mathcal{X}$ through $\mathcal{X} := \mathcal{S}^{-1}\mathcal{R}$. If we left-multiply both inequalities in (7.23) with $\mathcal{S}^{-1}$ and right-multiply with $\mathcal{S}^{-T}$, we arrive at (7.22). This shows that the constructed controller leads to (7.13) and the proof cycle is complete.

Theorem 7.11 *There exist A_K, B_K, C_K, D_K that solve the output-feedback H_∞ problem (7.13) if and only if there exist X and Y that satisfy the two ARIs*

$$A^TX + XA + XB_1B_1^TX + C_1^TC_1 - C_2^TC_2 < 0, \tag{7.33}$$

$$AY + YA^T + YC_1^TC_1Y + B_1B_1^T - B_2B_2^T < 0, \tag{7.34}$$

and the coupling condition

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0. \tag{7.35}$$

Suppose that X and Y satisfy (7.33)-(7.35). Let U and V be square and non-singular matrices with $UV^T = I - XY$, and set $L := -X^{-1}C_2^T$, $F := -B_2^TY^{-1}$. Then A_K, B_K, C_K as defined in (7.32),(7.31) lead to (7.13).

Remarks. The necessity of the conditions (7.33)-(7.35) has been proved for a strictly proper controller only. In fact, the conclusion does not require this hypothesis. On the other hand, the controller as given in the construction is always strictly proper. In general, if $D_{11} = 0$, the existence of a proper controller solving the H_∞-problem implies the existence of a *strictly proper* controller that solves the problem. Note, however, that one does in general not get a strictly proper controller that solves the H_∞-problem by simply removing the direct feedthrough term from a non-proper controller that solves the H_∞-problem!

Remark. If we recall (7.14), the formulas for the controller matrices can be simplified to

$$\begin{aligned} A_K &= -U^{-1}[A^T + XAY + X(B_1B_1^T - B_2B_2^T) + (C_1^TC_1 - C_2^TC_2)Y]V^{-T} \\ B_K &= -U^{-1}C_2^T, \\ C_K &= -B_2^TV^{-T}. \end{aligned} \tag{7.36}$$

For the particular choices $U = X$, $V = X^{-1} - Y$ or $U = Y^{-1} - X$, $V = Y$, one arrives at the specific formulas given in the literature. Note that all these constructions lead to controllers that have the *same order* as the generalized plant: the dimension of A_c is equal to the dimension of A.

We note again that (7.34)-(7.35) are equivalent to

$$\begin{pmatrix} A^TX + XA + C_1^TC_1 - C_2^TC_2 & XB_1 \\ B_1^TX & -I \end{pmatrix} < 0$$

and

$$\begin{pmatrix} AY + YA^T + B_1B_1^T - B_2B_2^T & YC_1^T \\ C_1Y & -I \end{pmatrix} < 0.$$

Hence testing the existence of solutions X and Y of (7.33)-(7.35) can be reduced to verifying the solvability of a system of linear matrix inequalities. In fact, numerical solvers provide solutions X, Y, and we have shown how to construct on the basis of these matrices a controller that satisfies (7.13).

7.8.2 Solution in Terms of Riccati Equations

As an alternative, Theorem 7.1 allows to go back to Riccati equations if $(A, B_1 B_1^T)$ and $(A^T, C_1^T C_1)$ are controllable as described in the following Riccati equation solution of the H_∞ problem.

Theorem 7.12 *Suppose that*

$$(A, B_1) \ \text{ is controllable and } \ (A, C_1) \ \text{ is observable.} \tag{7.37}$$

There exist A_K, B_K, C_K, D_K that solves the output-feedback H_∞ problem (7.13) if and only if the Riccati equations

$$A^T X + XA + XB_1 B_1^T X + C_1^T C_1 - C_2^T C_2 = 0, \tag{7.38}$$

$$AY + YA^T + YC_1^T C_1 Y + B_1 B_1^T - B_2 B_2^T = 0,$$

have anti-stabilizing solutions X_+, Y_+, and these solutions satisfy the coupling condition

$$\begin{pmatrix} X_+ & I \\ I & Y_+ \end{pmatrix} > 0. \tag{7.39}$$

With any non-singular U and V satisfying $XY + UV^T = I$, the formulas (7.36) define a controller that satisfies (7.13).

Note that the conditions in this result are algebraically verifiable without relying on linear matrix inequalities. One can test for the existence of anti-stabilizing solutions by verifying whether the corresponding Hamiltonian matrix has no eigenvalues on the imaginary axis. If the test is passed, one can compute these anti-stabilizing solutions. Then one simply needs to check (7.39) what amounts to verifying whether the smallest eigenvalue of the matrix on the left-hand side is positive.

Proof. Let A_K, B_K, C_K, D_K satisfy (7.13). By Theorem 7.11, there exist X, Y with (7.33)-(7.35). If we apply Theorem 7.1 twice, we infer that the AREs (7.38)-(7.12) have anti-stabilizing solutions X_+, Y_+ with $X \leq X_+$, $Y \leq Y_+$. The last two inequalities clearly imply (7.39).

Let us now suppose that X_+, Y_+ exist and satisfy (7.39), and construct the controller in the same fashion as described in Theorem 7.11. As easily seen, the matrix $\mathcal{X}$ we constructed above will satisfy

$$\mathcal{X} > 0, \quad \mathcal{A}^T\mathcal{X} + \mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X} + \mathcal{C}^T\mathcal{C} = 0. \tag{7.40}$$

Let us prove that $\mathcal{A}$ is stable. For that purpose we assume that $\mathcal{A}x = \lambda x$ with $x \neq 0$. If we left-multiply the ARE in (7.40) with x^* and right-multiply with x, we arrive at

$$\mathrm{Re}(\lambda)x^*\mathcal{X}x + x^*\mathcal{X}\mathcal{B}\mathcal{B}^T\mathcal{X}x + x^*\mathcal{C}^T\mathcal{C}x = 0.$$

Since $x^*\mathcal{X}x > 0$, we infer $\mathrm{Re}(\lambda) \leq 0$. Let us show that $\mathrm{Re}(\lambda) = 0$ cannot occur. In fact, if $\mathrm{Re}(\lambda) = 0$, we conclude

$$x^*\mathcal{X}\mathcal{B} = 0 \;\text{ and }\; \mathcal{C}x = 0.$$

Right-multiplying the ARE in (7.40) leads (with $-\overline{\lambda} = \lambda$) to

$$x^*\mathcal{X}\mathcal{A} = \lambda x^*\mathcal{X} \;\text{ and, still, }\; \mathcal{A}x = \lambda x$$

Set $\widehat{x} = \mathcal{X}x$, and partition $\widehat{x}$ and x according to $\mathcal{A}$. Then we arrive at

$$\begin{pmatrix} \widehat{y}^* & \widehat{z}^* \end{pmatrix} \begin{pmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{pmatrix} = \lambda \begin{pmatrix} \widehat{y}^* & \widehat{z}^* \end{pmatrix} \;\text{ and }\; \begin{pmatrix} \widehat{y}^* & \widehat{z}^* \end{pmatrix} \begin{pmatrix} B_1 \\ B_KD_{21} \end{pmatrix} = 0$$

as well as

$$\begin{pmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \lambda \begin{pmatrix} y \\ z \end{pmatrix} \;\text{ and }\; \begin{pmatrix} C_1 & D_{12}C_K \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0.$$

Right-multiplying with B_1 and left-multiplying with C_1^T imply that $\widehat{z}^*B_K = 0$ and $C_Kz = 0$. This reveals

$$\widehat{y}^*A = \lambda\widehat{y}^*,\; \widehat{y}^*B_1 = 0 \;\text{ and }\; Ay = \lambda y,\; C_1y = 0.$$

By controllability of (A, B_1) and observability of (A, C_1), we conclude $\widehat{y} = 0$ and $y = 0$. Finally,

$$\begin{pmatrix} 0 \\ \widehat{z} \end{pmatrix} = \begin{pmatrix} X_+ & U \\ U^T & * \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix} \;\text{ and }\; \begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} Y_+ & V \\ V^T & * \end{pmatrix} \begin{pmatrix} 0 \\ \widehat{z} \end{pmatrix}$$

lead to $0 = Uz$ and $0 = V\widehat{z}$ what implies $z = 0$ and $\widehat{z} = 0$ since U and V are non-singular. This is a contradiction to $x \neq 0$.

Since $\mathcal{A}$ is stable, the controller is stabilizing. Moreover, it is not difficult to verify that the Riccati equation in (7.40) implies $\|\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B}\|_\infty \leq 1$. One can in fact show that the strict inequality holds true; since this is cumbersome and not really relevant for our considerations, we stop here.

■

7.8.3 Solution in Terms of Indefinite Riccati Equations

We observe that $Y > 0$ and

$$AY + YA^T + YC_1^TC_1Y + B_1B_1^T - B_2B_2^T = 0, \quad \lambda(A + YC_1^TC_1) \subset \mathbb{C}_> \tag{7.41}$$

is equivalent to $Y_\infty > 0$ and

$$A^TY_\infty + Y_\infty A + Y_\infty(B_1B_1^T - B_2B_2^T)Y_\infty + C_1^TC_1 = 0,\ \lambda(A + (B_1B_1^T - B_2B_2^T)Y_\infty) \subset \mathbb{C}_< \tag{7.42}$$

with

$$Y_\infty = Y^{-1}.$$

Proof. Let us show that (7.41) implies (7.42). Left- and right-multiplying the ARE in (7.41) with $Y_\infty = Y^{-1}$ leads to the ARE in (7.42). Moreover, $Y > 0$ implies $Y_\infty = Y^{-1} > 0$. Finally, the ARE in (7.41) is easily rewritten to

$$(A + YC_1^TC_1)Y + Y(A^T + Y^{-1}[B_1B_1^T - B_2B_2^T]) = 0$$

what leads to

$$Y^{-1}(A + YC_1^TC_1)Y = -(A + [B_1B_1^T - B_2B_2^T]Y_\infty)^T.$$

Since $A + YC_1^TC_1$ is anti-stable, $A + [B_1B_1^T - B_2B_2^T]Y_\infty$ must be stable.

The converse implication (7.42) $\Rightarrow$ (7.41) follows by reversing the arguments.

■

Dually, we have $X > 0$ and

$$A^TX + XA + XB_1B_1^TX + C_1^TC_1 - C_2^TC_2 = 0, \quad \lambda(A + B_1B_1^TX) \subset \mathbb{C}_>$$

if and only if $X_\infty > 0$ and

$$AX_\infty + X_\infty A^T + X_\infty(C_1^TC_1 - C_2^TC_2)X_\infty + B_1B_1^T = 0,\ \lambda(A + X_\infty(C_1^TC_1 - C_2^TC_2)) \subset \mathbb{C}_<$$

with

$$X_\infty = X^{-1}.$$

Finally,

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0$$

is equivalent to

$$X_\infty > 0, \quad Y_\infty > 0, \quad \rho(X_\infty Y_\infty) < 1$$

with $X_\infty = X^{-1}$, $Y_\infty = Y^{-1}$.

Proof. The coupling condition is equivalent to $X > 0$, $Y > 0$, $Y - X^{-1} > 0$ (Schur) what is nothing but $X > 0$, $Y > 0$, $I - Y^{-1/2}X^{-1}Y^{-1/2} > 0$ or, equivalently, $X > 0$, $Y > 0$, $\rho(Y^{-1/2}X^{-1}Y^{-1/2}) < 1$ what can be rewritten (since $\rho(AB) = \rho(BA)$) as $X > 0$, $Y > 0$, $\rho(X^{-1}Y^{-1}) < 1$.

■

Hence, all conditions in Theorem 7.12 can be rewritten in terms of so-called indefinite algebraic Riccati equations. They are called indefinite since the matrices defining the quadratic terms are, in general, not positive or negative semi-definite.

For the particular choice of $U = Y^{-1} - X$, $V = Y$, the formulas for the controller can be rewritten in terms of $X_\infty = X^{-1}$, $Y_\infty = Y^{-1}$ as follows:

$$A_K = A - (I - X_\infty Y_\infty)^{-1} X_\infty C_2^T C_2 + (B_1 B_1^T - B_2 B_2^T) Y_\infty,$$
$$B_K = (I - X_\infty Y_\infty)^{-1} X_\infty C_2^T, \quad C_K = -B_2^T Y_\infty.$$

Proof. If we left-multiply (7.41) by Y^{-1}, we obtain

$$A^T + C_1^T C_1 Y = -[Y^{-1} A Y + Y^{-1}(B_1 B_1^T - B_2 B_2^T)].$$

Hence the formula for A_K in (7.36) can be rewritten to

$$A_K = -U^{-1}[(X - Y^{-1})AY - C_2^T C_2 Y + (X - Y^{-1})(B_1 B_1^T - B_2 B_2^T)]V^{-T}$$

what is nothing but

$$A_K = A + (Y^{-1} - X)^{-1} C_2^T C_2 + (B_1 B_1^T - B_2 B_2^T) Y^{-1}.$$

or

$$A_K = A - (I - X^{-1}Y^{-1})^{-1} X^{-1} C_2^T C_2 + (B_1 B_1^T - B_2 B_2^T) Y^{-1}.$$

The formulas for B_K and C_K are obvious.

■

Why are the results in the literature usually formulated in terms of these indefinite AREs? The simple reason is the possibility to relax the artificial and strong hypotheses that (A, B_1) and (A, C_1) are controllable and observable to a condition on the non-existence of uncontrollable or unobservable modes on the imaginary axis. The exact formulation with a general bound γ and with an explicit controller formula is as follows.

Theorem 7.13 *Suppose that*

$$(A, B_1), (A, C_1) \ \textit{have no uncontrollable, unobservable modes in } \mathbb{C}_=. \tag{7.43}$$

Then there exist A_K, B_K, C_K, D_K that solves the output-feedback H_∞ problem (7.13) if and only if the unique X_∞ and Y_∞ with

$$AX_\infty + X_\infty A^T + X_\infty(\frac{1}{\gamma^2}C_1^TC_1 - C_2^TC_2)X_\infty + B_1B_1^T = 0,$$
$$\lambda(A + X_\infty(\frac{1}{\gamma^2}C_1^TC_1 - C_2^TC_2)) \subset \mathbb{C}_<$$

and

$$A^TY_\infty + Y_\infty A + Y_\infty(\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T)Y_\infty + C_1^TC_1 = 0,$$
$$\lambda(A + (\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T)Y_\infty) \subset \mathbb{C}_<$$

exist, and they satisfy

$$X_\infty \geq 0, \quad Y_\infty \geq 0, \quad \rho(X_\infty Y_\infty) < \gamma^2.$$

If X_∞ and Y_∞ satisfy all these conditions, a controller with (7.13) is given by

$$\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = \left[\begin{array}{c|c} A - ZX_\infty C_2^TC_2 + \left[\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T\right]Y_\infty & ZX_\infty C_2^T \\ \hline -B_2^TY_\infty & 0 \end{array}\right]$$

where $Z = (I - \frac{1}{\gamma^2}X_\infty Y_\infty)^{-1}$.

Remark 7.14

1) The constructed controller in Theorem 7.13 is also called central controller. This terminology will be explained in Section 7.11.

2) X_∞ and Y_∞ are computed as for standard Riccati equations on the basis of the Hamiltonian matrices

$$H_{X_\infty} = \begin{pmatrix} A^T & \frac{1}{\gamma^2}C_1^TC_1 - C_2^TC_2 \\ -B_1B_1^T & -A \end{pmatrix} \quad \text{and} \quad H_{Y_\infty} = \begin{pmatrix} A & \frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{pmatrix}$$

as follows: Verify that H_{X_∞} and H_{Y_∞} do not have eigenvalues on the imaginary axis. Then compute (with Schur decompositions) $\begin{pmatrix} U_{X_\infty} \\ V_{X_\infty} \end{pmatrix}$, $\begin{pmatrix} U_{Y_\infty} \\ V_{Y_\infty} \end{pmatrix}$ and stable M_{X_∞}, M_{Y_∞} satisfying

$$H_{X_\infty}\begin{pmatrix} U_{X_\infty} \\ V_{X_\infty} \end{pmatrix} = \begin{pmatrix} U_{X_\infty} \\ V_{X_\infty} \end{pmatrix} M_{X_\infty} \quad \text{and} \quad H_{Y_\infty}\begin{pmatrix} U_{Y_\infty} \\ V_{Y_\infty} \end{pmatrix} = \begin{pmatrix} U_{Y_\infty} \\ V_{Y_\infty} \end{pmatrix} M_{Y_\infty}.$$

Verify that U_{X_∞} and U_{Y_∞} are non-singular. Then

$$X_\infty = V_{X_\infty} U_{X_\infty}^{-1} \quad \text{and} \quad Y_\infty = V_{Y_\infty} U_{Y_\infty}^{-1}$$

exist and are the stabilizing solutions of the two indefinite AREs under considerations.

After having verified that (the unique) X_∞ and Y_∞ exist, it remains to check whether they are both positive semi-definite, and whether the spectral radius of $X_\infty Y_\infty$ is smaller than γ^2.

2) Note that X_∞ and Y_∞ are, in general, not invertible. That's why we insisted on deriving formulas in which no inverse of X_∞ or Y_∞ occurs. If X_∞ and Y_∞ exist, one can show:

   X_∞ has no kernel if and only if (A, B_1) has no uncontrollable modes in the open left-half plane, and Y_∞ has no kernel if and only if (A, C_1) has no unobservable modes in the open left-half plane.

3) The optimal value
$$\gamma_* = \inf_{K \text{ stabilizies } P} \|S(P,K)\|_\infty$$
   is the smallest of all γ for which the stabilizing solutions X_∞, Y_∞ of the indefinite AREs exist and satisfy $X_\infty \geq 0$, $Y_\infty \geq 0$, $\rho(X_\infty Y_\infty) < \gamma^2$. The optimal value γ_* can hence be computed by bisection.

4) If $\gamma \leq \gamma_*$, it cannot be said a priori which of the conditions (existence of X_∞, Y_∞, positive semi-definiteness, coupling condition) fails.

   For $\gamma > \gamma_*$, but γ close to γ_*, it often happens that

$$I - \frac{1}{\gamma^2} X_\infty Y_\infty \ \text{ is close to singular.}$$

   Hence computing the inverse of this matrix is ill-conditioned. This leads to an ill-conditioned computation of the controller matrices as given in the theorem. Therefore, it is advisable not to get too close to the optimum.

   Note, however, that this is only due to the specific choice of the controller formulas. Under our hypothesis, one can show that there always exists an *optimal* controller. Hence there is a possibility, even $\gamma = \gamma_*$, to compute an optimal controller a well-conditioned fashion.

5) Let us consider the other extreme $\gamma = \infty$. Then one can *always find* $X_\infty \geq 0$ and $Y_\infty \geq 0$ that satisfy

$$AX_\infty + X_\infty A^T - X_\infty C_2^T C_2 X_\infty + B_1 B_1^T = 0, \ \lambda(A - X_\infty C_2^T C_2) \subset \mathbb{C}_<$$

and

$$A^T Y_\infty + Y_\infty A - Y_\infty B_2 B_2^T Y_\infty + C_1^T C_1 = 0,\ \ \lambda(A - B_2 B_2^T Y_\infty) \subset \mathbb{C}_<.$$

(Why?) Moreover, the controller formulas read as

$$\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = \left[\begin{array}{c|c} A - X_\infty C_2^T C_2 - B_2 B_2^T Y_\infty & X_\infty C_2^T \\ \hline -B_2^T Y_\infty & 0 \end{array}\right].$$

Clearly, this controller stabilizes P. In addition, however, it has even the additional property that it minimizes

$$\|S(P,K)\|_2 \ \text{ among all controllers } K \text{ which stabilize } P.$$

Here, $\|M\|_2$ is the so-called H_2-norm of the strictly proper and stable matrix M which is defined via

$$\|M\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}(M(i\omega)^* M(i\omega))\, d\omega.$$

Hence the controller is a solution to the so-called H_2-control problem. Since the H_2-norm can be seen to be identical to the criterion in LQG-control, we have recovered the controller that solves the LQG problem as it is taught in an elementary course. All this will be discussed in more detail and generality in the LMI course.

7.9 What are the Weakest Hypotheses for the Riccati Solution?

The command `hinfys` of the μ-tools to design an H_∞-controller requires the following hypotheses for the system

$$\begin{pmatrix} z \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \begin{pmatrix} w \\ u \end{pmatrix}$$

describing the interconnection:

- (A, B_2) is stabilizable, and (A, C_2) is detectable.
- D_{21} has full row rank, and D_{12} has full column rank.
- For all $\omega \in \mathbb{R}$, $\begin{pmatrix} A - i\omega I & B_1 \\ C_2 & D_{21} \end{pmatrix}$ has full row rank, and $\begin{pmatrix} A - i\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix}$ has full column rank.

If not true, the second and the third hypotheses can be easily enforced as follows: With some $\epsilon > 0$, solve the problem for the perturbed matrices

$$\begin{pmatrix} C_1 & D_{12} \end{pmatrix} \to \begin{pmatrix} C_1 & D_{12} \\ \epsilon I & 0 \\ 0 & \epsilon I \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} \to \begin{pmatrix} B_1 & \epsilon I & 0 \\ D_{21} & 0 & \epsilon I \end{pmatrix}, \quad D_{11} \to \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us denote the resulting generalized plant by

$$\begin{pmatrix} z \\ z_1 \\ z_2 \\ y \end{pmatrix} = P_\epsilon \begin{pmatrix} w \\ w_1 \\ w_2 \\ u \end{pmatrix} = \left[\begin{array}{c|cccc} A & B_1 & \epsilon I & 0 & B_2 \\ \hline C_1 & D_{11} & 0 & 0 & D_{12} \\ \epsilon I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon I \\ C_2 & D_{21} & 0 & \epsilon I & D_{22} \end{array}\right] \begin{pmatrix} w \\ w_1 \\ w_2 \\ u \end{pmatrix}.$$

The perturbation just amounts to introducing new disturbance signals w_1, w_2 and new controlled signals z_1, z_2 in order to render all the hypotheses for $\epsilon \neq 0$ satisfied.

Here are the precise conclusions that can be drawn for the relation of the H_∞ problem for the original interconnection P and for the perturbed interconnection P_ϵ.

- K stabilizes P if and only if K stabilizes P_ϵ. (Why?)
- For any K which stabilizes P and P_ϵ, the gain-interpretation of the H_∞ immediately reveals that
$$\|S(P,K)\|_\infty \leq \|S(P_\epsilon,K)\|_\infty.$$
Hence, if the controller K stabilizes P_ϵ and achieves $\|S(P_\epsilon,K)\|_\infty < \gamma$, then the very same controller also stabilizes P and achieves $\|S(P,K)\|_\infty < \gamma$. **This property does not depend on the size of $\epsilon > 0$!**
- For any K stabilizing P and P_ϵ, one has
$$\|S(P_\epsilon,K)\|_\infty \to \|S(P,K)\|_\infty \text{ for } \epsilon \to 0.$$
Hence, if there exists a K stabilizing P and rendering $\|S(P,K)\|_\infty < \gamma$ satisfied, the very same K stabilizes P_ϵ and achieves $\|S(P_\epsilon,K)\|_\infty < \gamma$ for some **sufficiently small** $\epsilon > 0$.

Note that there are other schemes to perturb the matrices in order to render the hypotheses satisfied. In general, however, it might not be guaranteed that the first of these two properties is true irrespective of the size of $\epsilon > 0$.

7.10 Game-Theoretic Interpretation

This more mathematically orientated chapter gives an extra interpretation of the H_∞ control problem in terms of game theory. The main result will be that optimal H_∞ control is the optimal strategy for a player who plays control stragies and tries to minimize a cost function. Optimal in the following sense: no matter what strategy its opponent plays, its pay-off is bounded from above by an optimal value and a deviation from the optimal strategy will leave its opponent the opportunity to drive the cost above the optimal value.

Let us now become more precise: We consider again the system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \quad x(0) = \xi \\ z &= C_1 x + D_{12} u \end{aligned} \tag{SY}$$

as in the state-feedback case under the following assumptions:

Hypothesis 7.15 *Suppose that*

$$(A, B_2) \text{ is stabilizable.} \tag{SF1}$$

$$D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}. \tag{SF2}$$

$$(A, C_1) \text{ is observable.} \tag{SF3}$$

Moreover let $Y_\infty \geq 0$ satisfy as in Theorem 7.13

$$A^T Y_\infty + Y_\infty A + Y_\infty(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) Y_\infty + C_1^T C_1 = 0, \tag{7.44}$$

$$\operatorname{eig}(A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) Y_\infty) \subset \mathbb{C}^- \tag{7.45}$$

with $\gamma > 0$.

Note that observability of (A, C_1) implies $Y_\infty > 0$. In linear quadratic game-theory, one considers the control u and the generalized disturbance w to be manipulated by adversary "players". Here the u-player tries to minimize and the w-player tries to maximize the quadratic cost (or pay-off)

$$\int_0^\infty \|z(t)\|^2 - \gamma^2 \|w(t)\|^2 dt. \tag{CO}$$

Our intention is to touch upon a specific consequence from the existence of Y_∞ - this is a tip of a whole iceberg of game-theory.

The first result is the following fact.

Lemma 7.16 *Suppose that Hypothesis 7.15 holds. If controlling (SY) with $\xi = 0$ as $u = -B_2^T Y_\infty x$ then*

$$\int_0^\infty \|z(t)\|^2 - \gamma^2 \|w(t)\|^2 dt \leq 0 \tag{7.46}$$

for all $w \in L_2$.

Proof. Since $Y_\infty > 0$ we can define $Y_+ := Y_\infty^{-1}$. Then Y_+ is the anti-stabilizing solution of the ARE

$$AY_+ + Y_+A + \gamma^{-2}B_1B_1^T - B_2B_2^T + Y_+C_1^TC_1Y_+ = 0$$

since

$$\begin{aligned} A^T + C_1^TC_1Y_+ &= Y_\infty(Y_+A^T + Y_+C_1^TC_1Y_+) \\ &= Y_\infty(-A^-\gamma^{-2}B_1B_1^TY_\infty + B_2B_2^TY_\infty)Y_\infty^{-1} \\ &= -Y_\infty(A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)Y_\infty)Y_\infty^{-1}. \end{aligned}$$

By Theorem 7.9 we can conclude that $F = -B_2^TY_\infty$ satisfies

$$\mathrm{eig}(A + B_2F) \subset \mathbb{C}^- \text{ and } \|(C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_1\| < \gamma.$$

Let $G(s)$ be the transfermatrix of the System (SY) controlled by $u = Fx$. Then $G(s) = (C_1 + D_{12}F)(sI - A - B_2F)^{-1}B_1$ and by using Parceval's identity

$$\int_0^\infty \|f(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{f}(i\omega)\|^2 d\omega \text{ for all } f \in L_2$$

we obtain

$$\begin{aligned} \int_0^\infty \|z(t)\|^2 dt &= \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{z}(i\omega)\|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \|G(i\omega)\hat{w}(i\omega)\|^2 d\omega \\ &\leq \|G\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{w}(i\omega)\|^2 d\omega \\ &\leq \gamma^2 \int_0^\infty \|w(t)\|^2 dt \end{aligned}$$

for all $w \in L_2$.

■

From a game-theoretic point of view Lemma 7.16 means that if the u-player plays the static-feedback strategy $u = -B_2^TY_\infty x$ for the System (SY) with $\xi = 0$, the cost (CO) is guaranteed to be bounded by 0 for any finite energy open-loop strategy the w-player uses.

As a next step we want to generalize this result for arbitrary initial condition ξ.

Lemma 7.17 *If Y_∞ satisfies only (7.44) then along any trajectory of (SY) with (SF2) holds:*

$$\begin{aligned} &x(T)^TY_\infty x(T) - \xi^TY_\infty\xi + \int_0^T \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \\ &= \int_0^T \|u(t) + B_2^TY_\infty x(t)\|^2 - \gamma^2 \int_0^T \|w(t) - \gamma^{-2}B_1^TY_\infty x(t)\|^2 dt. \end{aligned}$$

Proof. (7.44) implies

$$
\begin{aligned}
\frac{d}{dt} x(t)^T Y_\infty x(t) &= \dot{x}(t)^T Y_\infty x(t) + x(t)^T Y_\infty \dot{x}(t) \\
&= x(t)^T \left[A^T Y_\infty + Y_\infty A\right] x(t) + 2x(t)^T Y_\infty B_1 w(t) + 2x(t)^T Y_\infty B_2 u(t) \\
&= - x(t)^T C_1^T C_1 x(t) - \gamma^{-2} x(t)^T Y_\infty B_1 B_1^T Y_\infty x(t) + x(t)^T Y_\infty B_2 B_2^T Y_\infty x(t) \\
&\quad + 2x(t)^T Y_\infty B_1 w(t) + 2x(t)^T Y_\infty B_2 u(t) \\
&= - x(t)^T C_1^T C_1 x(t) + \gamma^2 w(t)^T w(t) - u(t)^T u(t) \\
&\quad - \gamma^2 \|w(t) - \gamma^{-2} B_1^T Y_\infty x(t)\|^2 + \|u(t) + B_2^T Y_\infty x(t)\|^2 \\
&= - \|z(t)\|^2 + \gamma^2 \|w(t)\|^2 + \|u(t) + B_2^T Y_\infty x(t)\|^2 \\
&\quad - \gamma^2 \|w(t) \gamma^{-2} B_1^T Y_\infty x(t)\|^2.
\end{aligned}
$$

Integration over $[0, T]$ proves the result.

■

Lemma 7.18 *Suppose that Hypothesis 7.15 holds and suppose (SY) is controlled by $u = -B_2^T Y_\infty x$. Then we have for all $w \in L_2$:*

$$\int_0^\infty \|z(t)\|^2 - \gamma^2 \|w(t)\|^2 dt \leq \xi^T Y_\infty \xi.$$

Equality holds for the feedback strategy $w = \gamma^{-2} B_1^T Y_\infty x$.

Proof. As in the proof of Lemma 7.16 it follows that $\operatorname{eig}(A - B_2 B_2^T Y_\infty) \in \mathbb{C}^-$. Now choose $w \in L_2$ arbitrary. Then any trajectory of the controlled system solves $\dot{x}(t) = (A - B_2 B_2^T Y_\infty) x(t) + B_1 w(t)$ with $w \in L_2$. By using Young's inequality for convolutions we can conclude $x \in L_2$. Hence $\dot{x} \in L_2$ and thus $\lim_{T \to \infty} x(T) = 0$. Now Lemma 7.17 implies

$$\xi^T Y_\infty \xi - \int_0^\infty \|z(t)\|^2 - \gamma^2 \|w(t)\|^2 dt = \gamma^2 \int_0^\infty \|w(t) - \gamma^{-2} B_1^T Y_\infty x(t)\|^2 dt \geq 0.$$

Equality holds for $w = \gamma^{-2} B_1^T Y_\infty x$. If $w = \gamma^{-2} B_1^T Y_\infty x$ any trajectory of the controlled system solves $\dot{x}(t) = (A - B_2 B_2^T Y_\infty + \gamma^{-2} B_1 B_1^T Y_\infty) x(t)$. Due to (7.45) we can conclude $x \in L_2$ and hence $w \in L_2$.

■

If the u-player plays the static state-feedback strategy $u = -B_2^T Y_\infty x$, the incurred cost is guaranteed to be bounded by $\xi^T Y_\infty \xi$, no matter which finite energy open-loop strategy the w-player uses. The adversary w-player has the possibility to maximize its pay-off by using the feedback strategy $w = \gamma^{-2} B_1^T Y_\infty x$. For a particular ξ this translates into a finite energy open-loop disturbance, which is the worst case disturbance from the perspective of the u-player. The resulting cost equals $\xi^T Y_\infty \xi$.

Reversing the roles of w and u leads to an analogous insight.

Lemma 7.19 *Suppose that Hypothesis 7.15 holds and let (SY) be affected by the worst-case disturbance strategy $w = \gamma^{-2}B_1^T Y_\infty x$. For any $u \in L_2$ such that $x \in L_2$ we have*

$$\xi^T Y_\infty \xi \leq \int_0^\infty \|z(t)\|^2 - \gamma^2 \|w(t)\|^2 dt,$$

with equality for the feedback control $u = -B_2^T Y_\infty x$.

If the w-player plays $w = -\gamma^2 B_1^T Y_\infty x$, its guaranteed pay-off is $\xi^T Y_\infty \xi$ for all finite energy open-loop strategies of u that take care of stabilizing the state. The u-player can react with $u = -B_2^T Y_\infty x$ to push its cost down to the lowest possible level $\xi^T Y_\infty \xi$. The order of the decision-taking of the players is important in these conclusions. Note that the roles of the players are not symmetric, since u has to take care of stabilization.

Let us hence only consider static-state feedback strategies for player u.

Theorem 7.20 *Suppose that Hypothesis 7.15 holds. Then the following equation holds:*

$$\min_{u=Fx,\ \operatorname{eig}(A+B_2F)\subset\mathbb{C}^-} \sup_{w\in L_2} \int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt = \xi^T Y_\infty \xi.$$

If u plays the stabilizing strategy $u = -Fx$ we infer

$$\sup_{w\in L_2} \int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \geq \xi^T Y_\infty \xi,$$

with equality (as we already know) for $F_\infty = -B_2^T Y_\infty$. We conclude that it is rational for the u-player to stick to $u = F_\infty x$, since a deviation is guaranteed to be not beneficial - it even does offer the opportunity for player w to drive the cost for u to values above $\xi^T Y_\infty \xi$.

One can view this result as a time-domain interpretation of H_∞-control.

In order to proof this result, we need some preparations.

Lemma 7.21 *Suppose that* $\operatorname{eig}(A) \subset \mathbb{C}^+$*. For every $w \in L_2$ there exists a unique ξ, such that the solution of $\dot{x} = Ax + Bw$, $x(0) = \xi$ is square integrable.*

Proof. Choose $w \in L_2$ arbitrary. Then for any system trajectory we know that

$$e^{-At}x(t) = \xi + \int_0^t e^{-A\tau}Bw(\tau)d\tau \text{ for } t \geq 0. \tag{7.47}$$

Since $-A$ is Hurwitz, the function $\tau \mapsto e^{-A\tau}Bw(\tau)$ is integrable on $[0, \infty)$ by the Hölder inequality.

If $x \in L_2$ then $\dot{x} \in L_2$, since $w \in L_2$ by assumption. Thus we obtain $\lim_{t\to\infty} x(t) = 0$. From (7.47) we infer $\xi = -\int_0^\infty e^{-A\tau} Bw(\tau)d\tau$ which proves uniqueness.
The solution for this initial condition is given by

$$x(t) = -e^{At} \int_0^\infty e^{-A\tau} Bw(\tau)d\tau + \int_0^t e^{A(t-\tau)} Bw(\tau)d\tau = -\int_t^\infty e^{A(t-\tau)} Bw(\tau)d\tau.$$

Using again Young's inequality for convolutions leads to $x \in L_2$ and hence existence.

■

Lemma 7.22 *Suppose that Hypothesis 7.15 holds and let Y be the stabilizing solution of the standard ARE: $A^TY + YA - YB_2B_2^TY + C_1^TC_1 = 0$. Fix $w \in L_2$ and choose the unique response $v \in L_2$ of $\dot{v} = -(A - B_2B_2^TY)^Tv - YB_1w$. If the control function $u \in L_2$ for (SY) assures $x \in L_2$ then*

$$\int_0^\infty \|z(t)\|^2 dt \geq \xi^T Y\xi + 2v(0)^T\xi - \int_0^\infty v(t)^T B_2B_2^T v(t) - 2v(t)^T B_1 w(t)dt. \qquad (7.48)$$

Equality holds for $u(t,x) = -B_2^TYx - B_2^Tv(t)$.

The right hand side of the inequality only depends on ξ and on $w(\cdot)$, but not on the control function $u(\cdot)$ or the resulting system trajectory of (SY).
This solves an LQ problem for a system driven by L_2-disturbances $w(\cdot)$. Therefore observe that (7.48) for $\gamma > 0$ can be written as

$$\int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \geq \xi^TY\xi + 2v(0)^T\xi - \int_0^\infty \begin{pmatrix} v(t) \\ w(t) \end{pmatrix}^T \begin{pmatrix} B_2B_2^T & -B_1 \\ -B_1^T & \gamma^2 I \end{pmatrix} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} dt.$$

The optimal strategy requires full knowledge of $w(\cdot)$ and is non-causal. This just means that at time t, the control action cannot be determined based on $w|_{[0,t]}$ only. This is true since by the proof of Lemma 7.21 $v(\cdot)$ is given by

$$v(t) = -\int_t^\infty e^{-(A-B_2B_2^TY)^T(t-\tau)} YB_1w(\tau)d\tau \text{ for } t \geq 0.$$

Hence knowledge of $w(\cdot)$ on $[t,\infty)$ is required to calculate $v(t)$ for a fixed $t \geq 0$. Finally note that under our assumptions there always exists a stabilitzing solution Y of the ARE: $A^TY + YA - YB_2B_2^TY + C_1^TC_1 = 0$.

Proof. As in Lemma 7.17 the proof follows by completion-of-the-squares and by using the fact that Y solves the given ARE.

$$
\begin{aligned}
\frac{d}{dt}(x^TYx + 2v^Tx) &= \dot{x}^TYx + x^TY\dot{x} + 2\dot{v}^Tx + 2v^T\dot{x} \\
&= x^T\left[A^TY + YA\right]x + 2x^TYB_1w + 2x^TYB_2u \\
&\quad - 2v^T(A - B_2B_2^TY)x - 2w^TB_1^TYx + 2v^T(Ax + B_1w + B_2u) \\
&= -x^TC_1^TC_1x + x^TYB_2B_2^TYx + 2v^TB_2B_2^TYx + v^TB_2B_2^Tv \\
&\quad - v^TB_2B_2^Tv + 2u^TB_2^T(Yx + v) + 2v^TB_1w \\
&= -\|C_1x\|^2 - \|u\|^2 + u^Tu + (Yx + v)^TB_2B_2^T(Yx + v) \\
&\quad + 2u^TB_2^T(Yx + v) - v^TB_2B_2^Tv + 2v^TB_1w \\
&= -\|z\|^2 + \|u + B_2^T(Yx + v)\|^2 - v^TB_2B_2^Tv + 2v^TB_1w.
\end{aligned}
$$

Since $x(T) \to 0$, $v(T) \to 0$ for $T \to \infty$ the statement follows by integration over $[0, T]$ and taking the limit $T \to \infty$.

■

Observe again that (7.48) can be written as

$$
\begin{aligned}
\int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \geq \xi^TY\xi + 2v(0)^T \\
- \int_0^\infty v(t)^TB_2B_2^Tv(t) - 2v(t)^TB_1w(t) + \gamma^2w(t)^Tw(t)dt.
\end{aligned}
$$

In the next step we intend to maximize the right-hand side over all pairs (w, v) as in Lemma 7.22. For given $w \in L_2$ by the proof of Lemma 7.21 we know that $v(0) = \int_0^\infty e^{(A-B_2B_2^TY)^T\tau}YB_1w(\tau)d\tau$. Thus the set of all $v(0)$ for such pairs is equal to the controllable subspace of $(-(A - B_2B_2^TY)^T, -YB_1)$. Hence these pairs can as well be parameterized by taking any v_0 in the controllable subspace and choosing $w \in L_2$ which assures $v \in L_2$ for

$$
\dot{v} = -(A - B_2B_2^TY)^Tv - YB_1w, \quad v(0) = v_0. \tag{AUX}
$$

For notational simplicity let us assume that (AUX) is controllable, by Hautus this means that

$$
(A - B_2B_2^TY,\ YB_1B_1^TY) \text{ is observable.} \tag{OBS}
$$

Let us first fix v_0. Then we have to solve the problem of minimizing

$$
\begin{aligned}
&\int_0^\infty v(t)^TB_2B_2^Tv(t) - 2v(t)^TB_1w(t) + \gamma^2w(t)^Tw(t)dt \\
= &\int_0^\infty \begin{pmatrix} v(t) \\ w(t) \end{pmatrix}^T \begin{pmatrix} B_2B_2^T & -B_1 \\ -B_1^T & \gamma^2I \end{pmatrix} \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} dt
\end{aligned}
$$

over all $w \in L_2$ with $v \in L_2$ for solutions of (AUX).

This is actually a standard LQ problem with stability, but the cost function involves cross-terms. This is the reasons why we provide a direct solution based on the following ARE:

$$-(A - B_2B_2^TY)Z - Z(A - B_2B_2^TY)^T + B_2B_2^T - \gamma^{-2}(I + ZY)B_1B_1^T(I + YZ) = 0.$$

Lemma 7.23 *Let Z be the anti-stabilizing solution of the ARE*

$$\begin{aligned}(A - B_2B_2^TY + \gamma^{-2}B_1B_1^TY)Z + Z(A - B_2B_2^TY + \gamma^{-2}B_1B_1^TY)^T \\ +\gamma^{-2}ZYB_1B_1^TYZ + \gamma^{-2}B_1B_1^T - B_2B_2^T = 0.\end{aligned}$$

If $w \in L_2$ achieves $v \in L_2$ for (AUX), the following inequality holds:

$$\int_0^\infty v(t)^TB_2B_2^Tv(t) - 2v(t)^TB_1w(t) + \gamma^2w(t)^Tw(t)dt \geq v_0^TZv_0.$$

Equality is achieved with $w = \gamma^{-2}B_1^T(I + YZ)v$.

Proof. As before we use the fact that Z solves the ARE and complete the squares:

$$\begin{aligned}\frac{d}{dt}v^TZv &= \dot{v}^TZv + v^TZ\dot{v} \\ &= v^T[-(A - B_2B_2^TY)Z - Z(A - B_2B_2^TY)^T]v - 2v^TZYB_1w \\ &= v^T[-B_2B_2^T + \gamma^{-2}(I + ZY)B_1B_1^T(I + YZ)]v - 2v^TZYB_1w \\ &= -v^TB_2B_2^Tv + 2v^TB_1w - \gamma^2w^Tw - 2v^T(I + ZY)B_1w \\ &\quad + \gamma^2w^Tw + \gamma^{-2}v^T(I + ZY)B_1B_1^T(I + YZ)v \\ &= -v^TB_2B_2^Tv + 2v^TB_1w - \gamma^2w^Tw + \gamma^2\|w - \gamma^{-2}B_2^T(I + YZ)v\|^2.\end{aligned}$$

Integrating over $[0, T]$ and taking the limit $T \to \infty$ leads to

$$\begin{aligned}\int_0^\infty v(t)^TB_2B_2^Tv(t) - 2v(t)^TB_1w(t) + \gamma^2w(t)^Tw(t)dt \\ = v(0)^TZv(0) + \int_0^\infty \gamma^2\|w - \gamma^{-2}B_1^T(I + YZ)v\|^2dt\end{aligned}$$

since $\lim_{T\to\infty} v(T) = 0$ by assumption. Hence we obtain the desired inequality.
Equality is enforced with $w = \gamma^{-2}B_1^T(I + YZ)v$ which results in

$$\dot{v} = -[(A - B_2B_2^TY + \gamma^{-2}B_1B_1^TY)^T + \gamma^{-2}YB_1B_1^TYZ]v, \quad v(0) = v_0.$$

Since Z is the anti-stabilizing solution of the given ARE we get $\mathrm{eig}(-(A - B_2B_2^TY + \gamma^{-2}B_1B_1^TY)^T - \gamma^{-2}YB_1B_1^TYZ) \subset \mathbb{C}^-$ and hence $v \in L_2$. This also implies $w = \gamma^{-2}B_1^T(I + YZ)v \in L_2$.

■

Since the following fact is very useful and needed in the following proofs, we formulate it as a lemma.

Lemma 7.24 *Let $R = R^T$, $Q_1 = Q_1^T$, $Q_2 = Q_2^T$ and let X_1 and X_2 be symmetric solutions of*

$$A^T X_1 + X_1 A + X_1 R X_1 + Q_1 = 0$$
$$A^T X_2 + X_2 A + X_2 R X_2 + Q_2 = 0.$$

Then $\Delta := X_2 - X_1$ solves the ARE

$$(A + RX_1)^T \Delta + \Delta(A + RX_1) + \Delta R \Delta + Q_2 - Q_1 = 0$$

Proof. Direct calculation leads to

$$\begin{aligned}
Q_1 - Q_2 &= (A^T X_2 + X_2 A + X_2 R X_2) - (A^T X_1 + X_1 A + X_1 R X_1) \\
&= A^T \Delta + \Delta A + X_2 R X_2 - X_1 R X_1 + X_1 R X_2 - X_1 R X_2 \\
&= A^T \Delta + \Delta A + \Delta R X_2 + X_1 R \Delta + \Delta R X_1 - \Delta R X_1 \\
&= A^T \Delta + \Delta A + X_1 R \Delta + \Delta R X_1 + \Delta R \Delta \\
&= (A + RX_1)^T \Delta + \Delta(A + RX_1) + \Delta R \Delta.
\end{aligned}$$

■

The next Lemma shows a relation between the appearing ARE's. The proof does not need the assumptions (SF1) - (SF3).

Lemma 7.25 *Let Y be as in Lemma 7.22 and let $Y_\infty \geq 0$ satisfy (7.44)-(7.45). If (OBS) holds then Z in Lemma 7.23 exists, is positive definite and equals $Z = (Y_\infty - Y)^{-1}$.*

Proof. Set $\tilde{A} := (A + \gamma^{-2} B_1 B_1^T Y - B_2 B_2^T Y)$ and set $\Delta := Y_\infty - Y$. Applying Lemma 7.24 to (7.44) and

$$A^T Y + YA + Y(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) Y + C_1^T C_1 - \gamma^{-2} Y B_1 B_1^T Y = 0$$

leads to the equation

$$\tilde{A}^T \Delta + \Delta \tilde{A} + \Delta(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)\Delta + \gamma^{-2} Y B_1 B_1^T Y = 0.$$

Let $x \in \ker(\Delta)$. Then $\gamma^{-2} \|B_1^T Y x\|^2 = 0$ and hence $0 = \Delta \tilde{A} x = \Delta(A - B_2 B_2^T Y)x$. This means $(A - B_2 B_2^T Y)\ker(\Delta) \subset \ker(\Delta)$ and $\ker(\Delta) \subset \ker(Y B_1 B_1^T Y)$. By (OBS) we can conclude $\ker(\Delta) = \{0\}$.
Set $Q := \gamma^{-2} Y_\infty B_1 B_1^T Y_\infty + C_1^T C_1$. Then observe that

$$0 = A^T Y_\infty + Y_\infty A - Y_\infty B_2 B_2^T Y_\infty + Q \tag{7.49}$$

and

$$A^T Y + YA - Y B_2 B_2^T Y + Q = \gamma^{-2} Y_\infty B_1 B_1^T Y_\infty \geq 0.$$

Choose $x \neq 0$ with $(A - B_2B_2^TY_\infty)x = \lambda x$. Then we obtain by using the ARE for Y_∞ :

$$\begin{aligned} 0 &= 2\mathrm{Re}(\lambda)\mathrm{x}^*\mathrm{Y}_\infty\mathrm{x} + \mathrm{x}^*\mathrm{Y}_\infty\mathrm{B}_2\mathrm{B}_2^\mathrm{T}\mathrm{Y}_\infty\mathrm{x} + \gamma^{-2}\mathrm{x}^*\mathrm{Y}_\infty\mathrm{B}_1\mathrm{B}_1^\mathrm{T}\mathrm{Y}_\infty\mathrm{x} + \mathrm{x}^*\mathrm{C}_1^\mathrm{T}\mathrm{C}_1\mathrm{x} \\ &= 2\mathrm{Re}(\lambda)\mathrm{x}^*\mathrm{Y}_\infty\mathrm{x} + \|\mathrm{B}_2^\mathrm{T}\mathrm{Y}_\infty\mathrm{x}\|^2 + \gamma^{-2}\|\mathrm{B}_1^\mathrm{T}\mathrm{Y}_\infty\mathrm{x}\|^2 + \|\mathrm{C}_1\mathrm{x}\|^2 \end{aligned}$$

Since $Y_\infty > 0$ and $\|B_2^TY_\infty x\|^2 + \gamma^{-2}\|B_1^TY_\infty x\|^2 + \|C_1x\|^2 \geq 0$ we obtain $\mathrm{Re}(\lambda) \leq 0$. Hence we obtain $\mathrm{eig}(A - B_2B_2^TY_\infty) \subset \mathbb{C}^- \cup \mathbb{C}^0$ (i.e. Y_∞ is the strong solution of the ARE (7.49)). By Exercise 6 we can conclude $Y \leq Y_\infty$ and hence $\Delta \geq 0$.
Now observe that $Z = \Delta^{-1} > 0$ satisfies the ARE in Lemma 7.23 and

$$-\Delta^{-1}[\tilde{A} + \gamma^{-2}ZYB_1B_1^TY]^T\Delta = \tilde{A} + (\gamma^{-2}B_1B_1^T - B_2B_2^T)\Delta = A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)Y_\infty. \tag{7.50}$$

Since $A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)Y_\infty$ is stable, $\tilde{A} + \gamma^{-2}ZYB_1B_1^TY$ is anti-stable.

■

We are now able to prove Theorem 7.20 under the additional assumption (OBS). If this is not true, one argues by reducing the dynamics (AUX) to the controllable subspace which is only notationally a bit more cumbersome.

Proof of Theorem 7.20. Take F with $\mathrm{eig}(A+BF) \subset \mathbb{C}^-$ and control (SY) with $u = Fx$. Due to Young's inequality for convolutions, for $w \in L_2$ the controlled system responds with $x \in L_2$ and we can define

$$\gamma_{wc}(F) := \sup_{w\in L_2} \int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \leq \infty.$$

For any $w \in L_2$ and $v \in L_2$ as in Lemma 7.22, we get

$$\begin{aligned} \gamma_{wc}(F) &\geq \int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \\ &\geq \xi^TY\xi + 2v(0)^T\xi - \int_0^\infty v(t)^TB_2B_2^Tv(t) - 2v(t)^TB_1w(t) + \gamma^2w(t)^Tw(t)dt. \end{aligned}$$

Let us now maximize the right-hand side over all such pairs (w, v). As remarked earlier, we can instead maximize over $w \in L_2$ for fixed $v(0) = v_0$ in the controllable subspace of $(-(A - B_2B_2^TY)^T, -YB_1)$ such that $v \in L_2$.
By Lemma 7.23 and Lemma 7.25 we obtain

$$\gamma_{wc}(F) \geq \xi^TY\xi + 2v_0^T\xi - v_0^TZv_0.$$

As a next step we maximize over v_0. Therefore observe that there exists v_* with $Zv_* = \xi$ and that Z has a positive definite square root. Hence

$$\begin{aligned} \gamma_{wc}(F) &\geq \xi^TY\xi + 2v_0^T\xi - v_0^TZv_0 \\ &= \xi^TY\xi + 2v_0^T\xi - v_0^TZv_0 + \xi^TZ^{-1}\xi - \xi^TZ^{-1}\xi \\ &= \xi^TY\xi + \xi^TZ^{-1}\xi - \|Z^{\frac{1}{2}}v_0 - Z^{-\frac{1}{2}}\xi\|^2. \end{aligned}$$

This shows that the maximum is achieved for $v_0 = Z^{-1}\xi = v_*$ and the maximum value of the right hand side is given by $\xi^T Y\xi + \xi^T Z^{-1}\xi = \xi^T Y_\infty \xi$ by Lemma 7.25. This proves

$$\min_{u=Fx,\ \mathrm{eig}(A+B_2F)\subset\mathbb{C}^-} \gamma_{wc}(F) \geq \xi^T Y_\infty \xi.$$

On the other hand, we know for $F_\infty = -B_2^T Y_\infty$ by Lemma 7.18 that

$$\min_{u=Fx,\ \mathrm{eig}(A+B_2F)\subset\mathbb{C}^-} \gamma_{wc}(F) \leq \gamma_{wc}(F_\infty) \leq \xi^T Y_\infty \xi.$$

This proves Theorem 7.20.

■

Let us now consider the situation that w plays with open-loop strategies $w \in L_2$, and that u responds with stabilizing open-loop strategies $u \in L_2$. The resulting pay-off is again given by the value $\xi^T Y_\infty \xi$.

Theorem 7.26 *Suppose that Hypothesis 7.15 holds. Then the following equation holds:*

$$\max_{w\in L_2} \min_{u\in L_2 \text{ such that } x\in L_2} \int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt = \xi^T Y_\infty \xi.$$

If the disturbance $w \in L_2$ acts on the system, the best choice for u leads to a cost not larger than $\xi^T Y_\infty \xi$ if u plays open-loop and takes care of stability. Moreover, the w-player can push the cost for u up to the level $\xi^T Y_\infty \xi$ by open-loop strategies $w \in L_2$

Proof. For $w \in L_2$ and $u \in L_2$ such that $x \in L_2$ define

$$\gamma(w,u) := \int_0^\infty \|z(t)\|^2 - \gamma\|w(t)\|^2 dt.$$

Let $w \in L_2$ be arbitrary. With $v \in L_2$ satisfying $\dot{v} = -(A - B_2B_2^T Y)^T v - YB_1 w$ and for the trajectory of $\dot{x} = (A - B_2B_2^T Y)x + B_1 w - B_2 B_2^T v$, $x(0) = \xi$ we infer from the equality in Lemma 7.22 and the inequality Lemma 7.23

$$\begin{aligned}
&\gamma(w, -B_2^T Y x - B_2^T v) \\
&= \xi^T Y\xi + 2v(0)^T\xi - \int_0^\infty v(t)^T B_2 B_2^T v(t) - 2v(t)^T B_1 w(t) + \gamma^2 w(t)^T w(t) dt \\
&\leq \xi^T Y\xi + 2v(0)^T\xi - v(0)^T Z v(0) \\
&= \xi^T Y\xi + \xi^T Z^{-1}\xi - \|Z^{1/2}v(0) - Z^{-1/2}\xi\|^2 \\
&\leq \xi^T Y_\infty \xi.
\end{aligned}$$

Hence we obtain

$$\max_{w\in L_2} \min_{u\in L_2 \text{ s.t. } x\in L_2} \gamma(w,u) \leq \xi^T Y_\infty \xi.$$

On the other hand choose $v(0) = v_*$ such that $v_* = Z^{-1}\xi$ and choose $w = \gamma^{-2}B_1^T(I+YZ)v$. Then using the inequality in Lemma 7.22 and the equality in Lemma 7.23 leads to

$$\begin{aligned}
&\max_{w\in L_2}\min_{u\in L_2\text{ s.t. }x\in L_2}\gamma(w,u)\\
&\geq \min_{u\in L_2\text{ s.t. }x\in L_2}\gamma(\gamma^{-2}B_1^T(I+YZ)v,u)\\
&\geq \xi^TY\xi + 2v_*^T\xi - \int_0^\infty v(t)^TB_2B_2^Tv(t) - 2v(t)^TB_1w(t) + \gamma^2w(t)^Tw(t)dt\\
&= \xi^TY\xi + 2v_*^T\xi + v_*^TZv_* = \xi^TY_\infty\xi.
\end{aligned}$$

This proves the claim.

■

The inner minimization of the u-player can be confined to stabilizing static state-feedback strategies without changing the value. This should be surprising in view of the structure of the optimal u-strategy for a fixed disturbance $w \in L_2$ as described in Lemma 7.22!

Theorem 7.27 *Suppose that Hypothesis 7.15 holds. Then the following equation holds:*

$$\begin{aligned}
\xi^TY_\infty\xi &= \min_{u=Fx,\,\mathrm{eig}(A+B_2F)\subset\mathbb{C}^-}\ \sup_{w\in L_2}\int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2dt\\
&= \sup_{w\in L_2}\ \inf_{u=Fx,\,\mathrm{eig}(A+B_2F)\subset\mathbb{C}^-}\int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2dt.
\end{aligned}$$

In game-theory, this means that the game with dynamics (SY) and with w playing finite energy open-loop strategies while u plays stabilizing static state-feedback strategies has a value, and the value of the game equals $\xi^TY_\infty\xi$.

Proof. The first equation is just Theorem 7.20. In order to shorten notation, define for $w \in L_2$ and $u = Fx$ with $\mathrm{eig}(A + B_2F) \in \mathbb{C}^-$

$$\zeta(w,u) := \int_0^\infty \|z(t)\|^2 - \gamma\|w(t)\|^2dt.$$

For $w \in L_2$ and $u = Fx$ with $\mathrm{eig}(A + B_2F) \in \mathbb{C}^-$ arbitrary we trivially get

$$\inf_{u=Fx,\,\mathrm{eig}(A+B_2F)\subset\mathbb{C}^-}\zeta(w,u) \leq \zeta(w,u).$$

And hence we obtain for arbitrary $u = Fx$ such that $\mathrm{eig}(A + B_2F) \subset \mathbb{C}^-$

$$\sup_{w\in L_2}\ \inf_{u=Fx,\,\mathrm{eig}(A+B_2F)\subset\mathbb{C}^-}\zeta(w,u) \leq \sup_{w\in L_2}\zeta(w,u).$$

With Theorem 7.20 this leads to

$$\sup_{w\in L_2}\ \inf_{u=Fx,\,\mathrm{eig}(A+B_2F)\subset\mathbb{C}^-}\zeta(w,u) \leq \inf_{u=Fx,\,\mathrm{eig}(A+B_2F)\subset\mathbb{C}^-}\ \sup_{w\in L_2}\zeta(w,u) = \xi^TY_\infty\xi.$$

Let $w \in L_2$ and choose F with $\mathrm{eig}(A+B_2F) \in \mathbb{C}^-$. Then the solution of $\dot{x} = (A+B_2F)x+B_1w$, $x(0) = \xi$ is in L_2, due to Young's inequality for convolutions. Hence $u = Fx \in L_2$ and

$$\inf_{u\in L_2 \text{ s.t. } x\in L_2} \int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \leq \inf_{u=Fx,\, \mathrm{eig}(A+B_2F)\subset\mathbb{C}^-} \zeta(w,u).$$

Taking the supremum over all $w \in L_2$ implies with Theorem 7.26

$$\xi^T Y_\infty \xi \leq \sup_{w\in L_2} \inf_{u=Fx,\, \mathrm{eig}(A+B_2F)\subset\mathbb{C}^-} \int_0^\infty \|z(t)\|^2 - \gamma^2\|w(t)\|^2 dt \leq \xi^T Y_\infty \xi$$

and hence equality.

■

Note that the optimal strategies in Theorem 7.26 are generated by solving

$$\begin{aligned} \dot{v} &= -(A - B_2B_2^TY + \gamma^{-2}(I+ZY)B_1B_1^TY)^T v, \quad v(0) = Z^{-1}\xi, \\ \dot{x} &= (A - B_2B_2^TY)x + (\gamma^{-2}B_1B_1^T(I+YZ) - B_2B_2^T)v, \quad x(0) = \xi. \end{aligned} \tag{7.51}$$

and with $u = -B_2^TYx - B_2^Tv$ as well as $w = \gamma^{-2}B_1^T(I+YZ)v$.
Moreover we can exploit (7.50) wich reads more explicitly as

$$-Z[A + \gamma^{-2}(I+ZY)B_1B_1^TY - B_2B_2^TY]^T Z^{-1} = A + \gamma^{-2}B_1B_1^TY_\infty - B_2B_2^TY_\infty.$$

The coordinate change $\eta = Zv$ in the dynamics of the optimal strategies (7.51) leads then to

$$\begin{aligned} \dot{\eta} &= (A + \gamma^{-2}B_1B_1^TY_\infty - B_2B_2^TY_\infty)\eta, \quad \eta(0) = \xi \\ \dot{x} &= Ax - B_2B_2^TY(x-\eta) + (\gamma^{-2}B_1B_1^TY_\infty - B_2B_2^TY_\infty)\eta, \quad x(0) = \xi. \end{aligned}$$

Since $\dot{x} - \dot{\eta} = (A - B_2B_2^TY)(x-\eta)$ and $x(0) - \eta(0) = 0$, we infer $x(t) = \eta(t)$ for all $t \geq 0$. Hence the worst-case disturbance and the optimal control are as well generated by static state-feedback as

$$\begin{aligned} w &= \gamma^{-2}B_1^T(I+YZ)v = \gamma^{-2}B_1^T(Z^{-1}+Y)\eta = \gamma^{-2}B_1^TY_\infty x, \\ u &= -B_2^TYx - B_2^Tv = -B_2^TYx - B_2^TZ^{-1}\eta = -B_2^TY_\infty. \end{aligned}$$

This fact will help in the next section to parameterize all controllers that solve the H_∞-problem.

7.11 Interpretation and Parametrization of Controllers

In this section we want to give an interpretation of the controller constructed in Theorem 7.13 which solves the output-feedback H_∞-problem.
Moreover we will parameterize all controllers that solve the output-feedback H_∞-problem. We will especially see that every such controllers equal $K_{\text{par}} \star Q$ for a fixed LTI system

K_{par} and a stable parameter Q with $\|Q\|_\infty < \gamma$.
In order to do so, we need again some preparations.

The following lemma reveals conditions under which external and internal stability are equivalent.

Lemma 7.28 *Let P and K be a generalized plant and a controller with the usual realizatios. Suppose that*

$$\begin{pmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \end{pmatrix} \text{ has full column rank for all } \lambda \in \mathbb{C}^0 \cup \mathbb{C}^+ \tag{7.52}$$

and

$$\begin{pmatrix} A-\lambda I & B_1 \\ C_2 & D_{21} \end{pmatrix} \text{ has full row rank for all } \lambda \in \mathbb{C}^0 \cup \mathbb{C}^+. \tag{7.53}$$

Then K stabilizes P iff $P \star K$ is stable.

If D_{12} has full column rank and $D_{12}^+ := (D_{12}^T D_{12})^{-1} D_{12}^T$ then (7.52) holds iff

$$(A - B_2 D_{12}^+ C_1, C_1 - D_{12} D_{12}^+ C_1) \text{ is detectable.}$$

If D_{21} has full row rank and $D_{21}^+ := D_{21}^T (D_{21} D_{21}^T)^{-1} D_{12}^T$ then (7.53) holds iff

$$(A - B_1 D_{21}^+ C_2, B_1 - B_1 D_{21}^+ D_{21}) \text{ is stabilizable.}$$

Also note that these properties are implied by controllability of (A, B_1), observability of (A, C_1) and

$$D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^T = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

The last property was just (7.14). It is an exercise (Exercise 2) to prove Lemma 7.28 and the properties above.

Definition 7.29 *A transfer matrix G is said to be* **all-pass** *if it has no poles on the imaginary axis and satisfies*

$$G(i\omega)^* G(i\omega) = I \text{ for all } \omega \in [0, \infty]. \tag{7.54}$$

G is said to be **inner** *if it is all-pass and stable.*

Analogously to inner transfer matrices $G \in RH_\infty$ is said to be **co-inner** if

$$G(i\omega)G(i\omega)^* = I \text{ for all } \omega \in [0, \infty].$$

Clearly a square transfer matrix G is inner if and only if G is co-inner. We will focus on inner transfer matrices. The properties for co-inner transfer matrices can be obtained by duality. Inner generalized plants P are very useful to us because of the properties in Lemma 7.32, which is a characterization of stabilizing controllers K that achieve $\|P \star K\|_\infty < 1$. They will appear when we modify the standard generalized plant with the help of the "worst-case disturbance" $-\gamma^{-2}B_1^T Y_\infty x$ and the optimal control $-B_2^T Y_\infty x$.

At first we present two characterizations of inner transfer matrices.

Lemma 7.30 *G is inner if and only if $\|Gu\|_2 = \|u\|_2$ for all $u \in L_2$.*

Proof. Suppose G is inner then also $G(i\omega)^*G(i\omega) = I$ for all $\omega \in \mathbb{R}$. Let $u \in L_2$ be arbitrary. Since G is stable $Gu \in L_2$ and we can use Parceval's identity:

$$\begin{aligned}
\|Gu\|_2^2 &= \int_0^\infty \|Gu(t)\|^2 dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(i\omega)\hat{u}(i\omega)\|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(i\omega)^* G(i\omega)^* G(i\omega)\hat{u}(i\omega) d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{u}(i\omega)\|^2 d\omega = \|u\|_2^2.
\end{aligned}$$

Now suppose $\|Gu\|_2 = \|u\|_2$ for all $u \in L_2$. Then $G \in RH_\infty$ and we get

$$\|G\|_\infty = \sup_{0<\|u\|_2<\infty} \frac{\|Gu\|_2}{\|u\|_2} = 1 \tag{7.55}$$

We now show that

$$\|G(i\omega)x\| = \|x\| \text{ for all } x \text{ and all } \omega \in \mathbb{R}.$$

Due to (7.55) we only need to show "$\geq$". Assume there exist ω_0 and x_0 with $\|G(i\omega_0)x_0\| < \|x_0\|$. Then the function $u(t) = e^{-t}(i\omega_0 + 1)x_0 \in L_2$ satisfies $\hat{u}(i\omega_0) = x_0$. We can now use Parceval's identity as above to obtain

$$0 = \int_{-\infty}^{\infty} \|\hat{u}(i\omega)\|^2 - \|G(i\omega)\hat{u}(i\omega)\|^2 d\omega$$

with $\|\hat{u}(i\omega)\|^2 - \|G(i\omega)\hat{u}(i\omega)\|^2 \geq 0$ for all $\omega \in \mathbb{R}$ and $\|\hat{u}(i\omega_0)\|^2 - \|G(i\omega_0)\hat{u}(i\omega_0)\|^2 > 0$. Since G is stable and due to continuity, this is a contradiction.

The polarisation identity now implies

$$\langle y, G(i\omega)^* G(i\omega)x\rangle = \langle y, x\rangle \text{ for all } x, y \text{ and all } \omega \in \mathbb{R}$$

and hence $G(i\omega)^*G(i\omega) = I$ for all $\omega \in [0,\infty]$.

$\blacksquare$

Theorem 7.31 *For $G(s) = C(sI - A)^{-1}B + D$ with $\mathrm{eig}(A) \subset \mathbb{C}^-$ let X denote the observability Gramian, the unique solution of the Lyapunov equation $A^TX + XA + C^TC = 0$. Then G is inner if*

$$B^TX + D^TC = 0 \text{ and } D^TD = I.$$

If (A,B) is controllable, the statement holds with "iff".

Proof. Define $G^*(s) := G(-s)^T$ to infer $G(i\omega)^* = G^*(i\omega)$. Note that

$$G^*(s) \text{ has the realization } \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right].$$

Therefore (7.54) holds iff the transfer matrix

$$G^*G = \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -A^T & C^TC & C^TD \\ 0 & A & B \\ \hline -B^T & D^TC & D^TD \end{array}\right]$$

equals the identity matrix I. A coordinate change with X implies that G^*G can also be realized as

$$\left[\begin{array}{cc|c} -A^T & A^TX + XA + C^TC & XB + C^TD \\ 0 & A & B \\ \hline -B^T & B^TX + D^TC & D^TD \end{array}\right] = H^* + H + D^TD$$

with $H(s) = (B^TX + D^TC)(sI - A)^{-1}B$ and $H^*(s) := H(-s)^T$. If $B^TX + D^TC = 0$ and $D^TD = I$ we get trivially $G^*(i\omega)G(i\omega) = I$ for all $\omega \in [0,\infty]$ and since G is stable by assumption, G is inner.

If G is inner, we can conclude that $H^*(i\omega) + H(i\omega)$ is constant for $\omega \in [0,\infty]$. Since H and H^* are strictly proper we obtain

$$0 = H^*(\infty) + H(\infty) = H^*(i\omega) + H(i\omega) = H(i\omega)^* + H(i\omega)$$

for all $\omega \in \mathbb{R}$. Hence $D^TD = I$.

As a next step we show that $0 = H(i\omega)^* + H(i\omega)$ for all $\omega \in \mathbb{R}$ implies $H = 0$. This requires some arguments from complex analysis which can be read-off in any book on complex analysis.

Choose arbitrary vectors u,v and set $\Omega := \mathbb{C}\backslash(\mathrm{eig}(A) \cup \mathrm{eig}(-A^T))$. Then define the analytic function $f : \Omega \to \mathbb{C}, f(z) = \langle u, H(z)v\rangle$ and $g : \Omega \to \mathbb{C}, g(z) = \langle u, -H(-z)^Tv\rangle$.

By assumption f equals g on $\mathbb{C}^0$ and hence f equals g on Ω. Since f is analytic on $\mathbb{C}^+$ and g is analytic on $\mathbb{C}^-$, f has an analytic continuation $\tilde{f}$ on $\mathbb{C}$. Strict properness of H implies $f(z) \to 0$ for $|z| \to \infty$ and the same holds for the continuation $\tilde{f}$. Hence we can conclude that $\tilde{f}$ is bounded on $\mathbb{C}$ and by Liouville $\tilde{f}$ is constant on $\mathbb{C}$. Using again that H is strict proper implies that $\tilde{f}$ vanishes on $\mathbb{C}$. Hence $\langle u, H(z)v\rangle = 0$ for all $z \in \mathbb{C}$ and all u, v which means $H = 0$.

Now assume there exists x_0 with $(B^T X + D^T C)x_0 \neq 0$. By controllability of (A, B) we find a smooth u such that the solution of $\dot{x} = Ax + Bu$ satisfies $x(1) = x_0$. We can make sure that $u \in L_2$. Then $y(1) = (B^T X + D^T C)x_0 \neq 0$. On the other hand we get with Parceval

$$\begin{aligned} \|y\|_2^2 &= \int_0^\infty \|Hu(t)\|^2 dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \|H(i\omega)\hat{u}(i\omega)\|^2 d\omega = 0. \end{aligned}$$

Since y is also continuous, this means $y(t) = 0$ for all $t \in [0, \infty)$. This is a contradiction.

■

The proof also shows that G is inner if and only if $G(-z)^T G(z) = I$ for all $z \in \mathbb{C}$.

If a generalized plant is inner and the transfer function from w to y has a stable inverse, stabilizing controllers that achieve an H_∞-norm bound 1 are very special: they must be contained in the open unit ball of RH_∞. The converse holds as well.

Lemma 7.32 *Consider the inner generalized plant*

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \quad \text{with } P_{21}^{-1} \in RH_\infty.$$

Then the transfer matrix K stabilizes P and achieves $\|P \star K\|_\infty < 1$ if and only if

$$K \in RH_\infty \text{ and } \|K\|_\infty < 1.$$

This result is instrumental for parameterizing all suboptimal controllers.

Proof. "$\Leftarrow$": Since P is inner, we can conclude that $\|P\|_\infty = 1$ and hence

$$\|P_{22}\|_\infty \leq \|P\|_\infty = 1.$$

We can apply Theorem 4.10 to obtain stability of $(I - P_{22}K)^{-1}$. Since K and P are also stable K internally stabilizes P.

Let $\|K\|_\infty \leq \gamma$ with $\gamma \in (0, 1)$ and define $\kappa = \|P_{21}^{-1}(I - P_{22}K)\|_\infty$. Now choose any $\omega \in [0, \infty]$ and any complex vector x. Then $z = (P \star K)(i\omega)x$ satisfies, with $y =$

$(I - P_{22}(i\omega)K(i\omega))^{-1}P_{21}(i\omega)x$ the relations

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} P_{11}(i\omega) & P_{12}(i\omega) \\ P_{21}(i\omega) & P_{22}(i\omega) \end{pmatrix} \begin{pmatrix} x \\ K(i\omega)y \end{pmatrix}. \tag{7.56}$$

Since $P(i\omega)^* P(i\omega) = I$, we infer

$$\|z\|^2 + \|y\|^2 = \|x\|^2 + \|K(i\omega)y\|^2 \le \|x\|^2 + \gamma^2 \|y\|^2.$$

Since $x = P_{21}(i\omega)^{-1}(I - P_{22}(i\omega)K(i\omega))y$ we have $\|x\|^2 \le \kappa^2 \|y\|^2$ and thus, due to $\gamma^2 - 1 < 0$ and $\kappa > 0$,

$$\|z\|^2 \le [1 + (\gamma^2 - 1)\kappa^{-2}]x\|^2.$$

Since x, ω were arbitrary, we get $\|(P \star K)(i\omega)\| < 1$ for all ω and hence $\|P \star K\|_\infty < 1$.

"$\Rightarrow$": Suppose $\sup_{\omega \in [0,\infty]} \|K(i\omega)\| \ge 1$. Then we can choose $\omega \in [0, \infty]$ such that $i\omega$ is not a pole of K and some complex vector $y \ne 0$ with $\|K(i\omega)y\| \ge \|y\|$. If defining $x := P_{21}(i\omega)^{-1}(I - P_{22}(i\omega)K(i\omega))y$ and $z := (P \star K)(i\omega)x$, the relation (7.56) holds again. This now implies

$$\|z\|^2 + \|y\|^2 = \|x\|^2 + \|K(i\omega)y\|^2 \ge \|w\|^2 + \|y\|^2$$

and thus $\|(P \star K)(i\omega)x\| = \|z\| \ge \|x\|$. We infer $\|(P \star K)(i\omega)\| \ge 1$, which in turn means $\|P \star K\|_\infty \ge 1$, a contradiction.

We conclude $\|K(i\omega)\| < 1$ for all $\omega \in [0, \infty]$. In particular, K has no poles in $\mathbb{C}^0$. Since P is inner, it trivially satisfies $\|P\|_\infty \le 1$. Since K stabilizes P we can conclude stability of K by using Lemma 4.6.

■

We now re-describe the generalized plant P. Therefore let P be as usual

$$\begin{aligned} \dot{x} &= A\ x + B_1\ w + B_2\ u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

under the following hypothesis.

Hypothesis 7.33 *Suppose that*

$$(A, B_2) \text{ is stabilizable and } (A, C_2) \text{ is detectable.} \tag{OF1}$$

$$D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^T = \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{OF2}$$

$$D_{11} = 0 \text{ and } D_{22} = 0. \tag{OF3}$$

$$(A, C_1) \text{ is observable and } (A, B_1) \text{ is controllable.} \tag{OF4}$$

Moreover suppose that there exist $Y_\infty \geq 0$ with

$$A^T Y_\infty + Y_\infty A + Y_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)Y_\infty + C_1^T C_1 = 0, \tag{7.57}$$
$$\operatorname{eig}(A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)Y_\infty) \subset \mathbb{C}^- \tag{7.58}$$

with $\gamma > 0$.

Lemma 7.17 motivates to introduce the new disturbance/control variable

$$d = w - \gamma^{-2}B_1^T Y_\infty x \text{ and } v = u - F_\infty x,\ F_\infty := -B_2^T Y_\infty \tag{7.59}$$

for the generalized plant P if ignoring y. Note that d is the deviations of the worst case disturbance $\gamma^{-2}B_1^T Y_\infty x$ and that v is the deviation of the optimal control $-B_2^T Y_\infty x$ in the sense of Lemma 7.18, Lemma 7.19 and Theorem 7.20. For $x(0) = 0$ and trajectories of finite energy, Lemma 7.17 implies with $T \to \infty$ that

$$\|z\|_2^2 + \gamma^2\|d\|_2^2 = \gamma^2\|w\|_2^2 + \|v\|_2^2.$$

This means that the following system is inner:

$$G: \begin{pmatrix} \dot{x} \\ \hline z \\ \gamma d \end{pmatrix} = \left(\begin{array}{c|cc} A + B_2F_\infty & \gamma^{-1}B_1 & B_2 \\ \hline C_1 + D_{12}F_\infty & 0 & D_{12} \\ -\gamma^{-1}B_1^T Y_\infty & I & 0 \end{array}\right) \begin{pmatrix} x \\ \hline \gamma w \\ v \end{pmatrix}. \tag{7.60}$$

It is indeed possible to algebraically verify the following result.

Lemma 7.34 *Suppose that Hypothesis 7.33 holds. Then G defined in (7.60) satisfies all hypothesis of Lemma 7.32.*

Proof. Let G denote the transfer matrix of (7.60) partioned accordingly. Since $A + B_2F_\infty + \gamma^{-2}B_1B_1^T Y_\infty$ is Hurwitz and (7.57) holds, we infer as already done $\operatorname{eig}(A + B_2F_\infty) \subset \mathbb{C}^-$ with Theorem 7.9 as in the proof of Lemma 7.16. Thus we get

$$G \in RH_\infty \text{ and } G_{21}^{-1} = \left[\begin{array}{c|c} A + B_2F_\infty + \gamma^{-2}B_1B_1^T Y_\infty & \gamma^{-1}B_1 \\ \hline \gamma^{-1}B_1^T Y_\infty & I \end{array}\right] \in RH_\infty.$$

Due to $(C_1 + D_{12}F_\infty)^T(C_1 + D_{12}F_\infty) = C_1^T C_1 + Y_\infty B_2 B_2^T Y_\infty$, the ARE for Y_∞ can be written as

$$(A + B_2F_\infty)^T Y_\infty + Y_\infty(A + B_2F_\infty) + \begin{pmatrix} C_1 + D_{12}F_\infty \\ -\gamma^{-1}B_1^T Y_\infty \end{pmatrix}^T \begin{pmatrix} C_1 + D_{12}F_\infty \\ -\gamma^{-1}B_1^T Y_\infty \end{pmatrix} = 0.$$

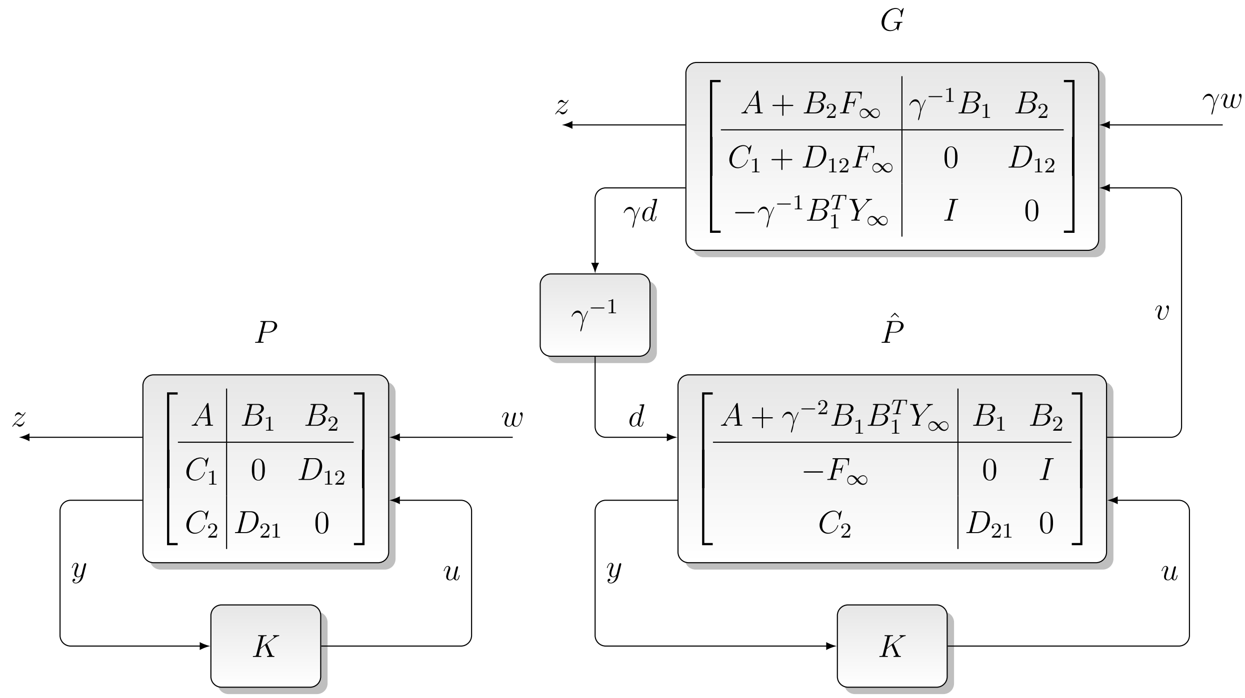


Figure 60: Re-description of the generalized plant P.

Hence Y_∞ is the observability Gramian corresponding to G. Moreover we trivially have

$$\begin{pmatrix} \gamma^{-1}B_1^T \\ B_2^T \end{pmatrix} Y_\infty + \begin{pmatrix} 0 & I \\ D_{12}^T & 0 \end{pmatrix} \begin{pmatrix} C_1 + D_{12}F_\infty \\ -\gamma^{-1}B_1^T Y_\infty \end{pmatrix} = 0$$

and $D_{12}^T D_{12} = I$. Hence G is inner by Theorem 7.31.

■

If we apply (7.59) to P while ignoring z, we optain

$$\hat{P}: \quad \begin{pmatrix} \dot{x} \\ \hline v \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A + \gamma^{-2}B_1B_1^T Y_\infty & B_1 & B_2 \\ \hline -F_\infty & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right) \begin{pmatrix} x \\ \hline d \\ u \end{pmatrix}. \tag{7.61}$$

By the mere definitions of the systems (7.60)-(7.61), their interconnection (both uncontrollend and controlled) as depicted in Figure 60 just constitutes another description of the original generalized plant P.

Similarly to P, G also satisfies the properties of Lemma 7.28. Then a direct application of Lemma 7.32 with G leads to the following key relation.

Lemma 7.35 *Suppose that Hypothesis 7.33 holds. Then the transfer matrix K stabilizes P with $\|P \star K\|_\infty < \gamma$ iff $\hat{P} \star K$ is stable with $\|\hat{P} \star K\|_\infty < \gamma$.*

Proof. Due to (OF2) and (OF4) P satisfies the hypothesis of Lemma 7.28. The same is true for G. Since

$$(A + B_2F_\infty - B_2D_{12}^+(C_1 + D_{12}F_\infty),\ C_1 + D_{12}F_\infty - D_{12}D_{12}^+(C_1 + D_{12}F_\infty)) = (A, C_1)$$

is detectable and

$$(A + B_2F_\infty + \gamma^{-2}B_1B_1^TY_\infty,\ 0)$$

is stabilizable.

Now suppose K stabilizes P with $\|P \star K\|_\infty < \gamma$. Then we get as depicted in Figure 60

$$(P \star K)\gamma^{-1} = G \star \hat{K} \tag{7.62}$$

for $\hat{K} := (\hat{P} \star K)\gamma^{-1}$. Hence $G \star \hat{K} \in RH_\infty$ with $\|G \star \hat{K}\|_\infty < 1$. By Lemma 7.28, $\hat{K}$ stabilizes G. Then Lemma 7.32 implies that $\hat{K}$ is stable and $\|\hat{K}\|_\infty < 1$. Hence $\hat{P} \star K$ is stable with $\|\hat{P} \star K\|_\infty < \gamma$.

Conversely, $\hat{P} \star K \in RH_\infty$ and $\|\hat{P} \star K\|_\infty < \gamma$ implies for $\hat{K} := (\hat{P} \star K)\gamma^{-1}$ that $\hat{K} \in RH_\infty$ and $\|\hat{K}\|_\infty < 1$. By Lemma 7.32, $\hat{K}$ stabilizes G with $\|G \star \hat{K}\|_\infty < 1$. By (7.62), $P \star K \in RH_\infty$ with $\|P \star K\|_\infty < \gamma$. By Lemma 7.28, K also stabilizes P.

■

As a next step we express the conditions in Theorem 7.13 as follows.

Lemma 7.36 *Suppose that (OF1) - (OF4) hold. Then there exist $X_\infty \geq 0$ and $Y_\infty \geq 0$ with*

$$\begin{aligned} AX_\infty + X_\infty A^T + X_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)X_\infty + B_1B_1^T &= 0 \\ \mathrm{eig}(A + X_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)) &\subset \mathbb{C}^- \\ A^TY_\infty + Y_\infty A + Y_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)Y_\infty + C_1^TC_1 &= 0 \\ \mathrm{eig}(A + Y_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)) &\subset \mathbb{C}^- \end{aligned}$$

and

$$\rho(X_\infty Y_\infty) < \gamma^2$$

if and only if there exists $Y_\infty \geq 0$ and $\hat{X}_\infty \geq 0$ with

$$\begin{aligned} A^TY_\infty + Y_\infty A + Y_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)Y_\infty + C_1^TC_1 &= 0 \\ \mathrm{eig}(A + Y_\infty(\gamma^{-2}B_1B_1^T - B_2B_2^T)) &\subset \mathbb{C}^- \end{aligned}$$

and, for $\hat{A} := A + \gamma^{-2}B_1B_1^TY_\infty$ and $F_\infty := -B_2^TY_\infty$,

$$\begin{aligned} \hat{A}\hat{X}_\infty + \hat{X}_\infty\hat{A}^T + \hat{X}_\infty(\gamma^{-2}F_\infty^TF_\infty - C_2^TC_2)\hat{X}_\infty + B_1B_1^T &= 0 \\ \mathrm{eig}(\hat{A} + \hat{X}_\infty(\gamma^{-2}F_\infty^TF_\infty - C_2^TC_2)) &\subset \mathbb{C}^-. \end{aligned}$$

The transformation is achieved with $\hat{X}_\infty = (X_\infty^{-1} - \gamma^{-2}Y_\infty)^{-1}$.

The two decoupled AREs with coupled solutions are thus rewritten into two coupled AREs whose solutions are not coupled any more. More interestingly, the ARE for $\hat{X}_\infty$ admits an important interpretation!

Proof. "$\Rightarrow$": Due to (OF4) X_∞ and Y_∞ are invertible. We can write the given ARE's as

$$A^T X_\infty^{-1} + X_\infty^{-1} A + X_\infty^{-1} B_1 B_1^T X_\infty^{-1} + \gamma^{-2} C_1^T C_1 - C_2^T C_2 = 0$$

and

$$A^T(\gamma^{-2} Y_\infty) + (\gamma^{-2} Y_\infty) A + (\gamma^{-2} Y_\infty) B_1 B_1^T (\gamma^{-2} Y_\infty) - \gamma^{-2} F_\infty^T F_\infty + \gamma^{-2} C_1^T C_1 = 0.$$

We can now apply Lemma 7.24 to $\Delta = X_\infty^{-1} - \gamma^{-2} Y_\infty$ and get

$$\hat{A}^T \Delta + \Delta \hat{A} + \Delta B_1 B_1^T \Delta + \gamma^{-2} F_\infty^T F_\infty - C_2^T C_2 = 0 \tag{7.63}$$

with $\hat{A} := A + \gamma^{-2} B_1 B_1^T Y_\infty$. Since Y_∞ is positive definite, it has a positive definite square root. Hence we can use $\gamma^2 > \rho(X_\infty Y_\infty) = \rho(Y_\infty^{1/2} X_\infty Y_\infty^{1/2})$ to obtain $\lambda_{\min}(Y_\infty^{-1/2} X_\infty^{-1} Y_\infty^{-1/2}) > \gamma^{-2}$ since X_∞ is invertible. This implies

$$\Delta = X_\infty^{-1} - \gamma^{-2} Y_\infty = Y_\infty^{1/2}(Y_\infty^{-1/2} X_\infty^{-1} Y_\infty^{-1/2} - \gamma^{-2} I) Y_\infty^{1/2} > 0.$$

Now observe that

$$\hat{A} + B_1 B_1^T \Delta = A + B_1 B_1^T X_\infty^{-1} = X_\infty(-A^T - (\gamma^{-2} C_1^T C_1 - C_2^T C_2) X_\infty) X_\infty^{-1}$$

is anti-stable. Hence we can conclude with $\hat{X}_\infty := \Delta^{-1} > 0$ that

$$\hat{A} + \hat{X}_\infty(\gamma^{-2} F_\infty^T F_\infty - C_2^T C_2) = \Delta^{-1}(-\hat{A}^T - \Delta B_1 B_1^T) \Delta$$

is Hurwitz. Together with (7.63) this completes "if".

"$\Leftarrow$": (OF4) implies again that Y_∞ and $\hat{X}_\infty$ are invertible since $(\hat{A}, B_1)$ is controllable. Then $\tilde{X}_\infty := \gamma^{-2} Y_\infty + \hat{X}_\infty^{-1} > 0$ solves the ARE

$$A^T \tilde{X}_\infty + \tilde{X}_\infty A + \tilde{X}_\infty B_1 B_1^T \tilde{X}_\infty + \gamma^{-2} C_1^T C_1 - C_2^T C_2 = 0$$

Now set $X_\infty := \tilde{X}_\infty^{-1} > 0$ to see that X_∞ solves the desired ARE. As before since $\mathrm{eig}(\hat{A} + \hat{X}_\infty(\gamma^{-2} F_\infty^T F_\infty - C_2^T C_2)) \subset \mathbb{C}^-$ we can conclude $\mathrm{eig}(A + X_\infty(\gamma^{-2} C_1^T C_1 - C_2^T C_2) \subset \mathbb{C}^-$. Finally observe that

$$0 < \hat{X}_\infty^{-1} = X_\infty^{-1} - \gamma^{-2} Y_\infty = Y_\infty^{1/2}(Y_\infty^{-1/2} X_\infty^{-1} Y_\infty^{-1/2} - \gamma^{-2} I) Y_\infty^{1/2}.$$

Since $Y_\infty^{1/2} > 0$, this implies $\lambda_{\min}(Y_\infty^{-1/2} X_\infty^{-1} Y_\infty^{-1/2}) > \gamma^{-2}$. Hence we can conclude $\gamma^2 > \rho(Y_\infty^{1/2} X_\infty Y_\infty^{1/2}) = \rho(X_\infty Y_\infty)$.

$\blacksquare$

7.11.1 Interpretation of the Output-Feedback Controller

Let us assume that there exists a K stabilizing P with $\|P \star K\|_\infty < \gamma$. By Theorem 7.13, we can construct the central controller. We are now ready to provide an interpretation of this controller and to prove that it really solves the H_∞-problem. By Lemma 7.35, the latter requires to show that the controller in Theorem 7.13 renders $\hat{P} \star K$ stable with $\|\hat{P} \star K\|_\infty < \gamma$.

The optimal state-feedback controller for $\hat{P}$ is $u = F_\infty x$ (with optimal value zero). If only y is measured, one could try to reconstruct the required control action with an H_∞-observer for $\hat{P}$ with the structure

$$\dot{\hat{x}} = \hat{A}\hat{x} + B_2 u + L(C_2\hat{x} - y), \;\; \hat{u} = F_\infty \hat{x},$$

and then control $\hat{P}$ with $u = \hat{u}$. Theorem 7.10 and Lemma 7.36 motivate to choose the observer gain $\hat{L}_\infty = -\hat{X}_\infty C_2^T$. This leads to the controller

$$\dot{\hat{x}} = (\hat{A} + \hat{L}_\infty C_2)\hat{x} + B_2 u - \hat{L}_\infty y, \;\; u = F_\infty \hat{x}. \tag{7.64}$$

Due to $\hat{X}_\infty = ZX_\infty = (I - \gamma^{-2}X_\infty Y_\infty)^{-1}X_\infty$, this is exactly the controller in Theorem 7.13.

If we introduce the "worst-case disturbance" into the original generalized plant by $w = d + \gamma^{-2}B_1^T Y_\infty x$, we obtain

$$\begin{pmatrix} \dot{x} \\ \hline z \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A + \gamma^{-2}B_1B_1^TY_\infty & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right) \begin{pmatrix} x \\ \hline d \\ u \end{pmatrix}.$$

This leads to the following interpretation of the central controller:
The output-feedback H_∞-controller results from replacing u in the H_∞-state-feedback controller $u = F_\infty x$ with the estimate $\hat{u}$, obtained by an H_∞-observer for the output $F_\infty x$ of the generalized plant "in the presence of the worst-case disturbance $\gamma^{-2}B_1^TY_\infty x$".

This is often called the separation principle in H_∞-theory. For $\gamma = \infty$, it reduces to the classical separation principle for H_2-controllers, since then the "worst-case disturbance" vanishes.

We still owe the proof that the constructed controller (7.64) based on the solutions of the Riccati equations does indeed solve the H_∞-problem.

Lemma 7.37 *Suppose that Hypothesis 7.33 holds and that there exists $X_\infty \geq 0$ with*

$$AX_\infty + X_\infty A^T + X_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)X_\infty + B_1B_1^T = 0$$
$$\mathrm{eig}(A + X_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)) \subset \mathbb{C}^-$$

and

$$\rho(X_\infty Y_\infty) < \gamma^2.$$

Moreover let $\hat{A} = A + \gamma^{-2} B_1 B_1^T$, $\hat{X}_\infty = (X_\infty^{-1} - \gamma^{-2} Y_\infty)^{-1}$, $\hat{L}_\infty = -\hat{X}_\infty C_2^T$ *and* $F_\infty = -B_2^T F_\infty$. *Then the controller* K *defined by*

$$\dot{\hat{x}} = (\hat{A} + \hat{L}_\infty C_2)\hat{x} + B_2 u - \hat{L}_\infty y, \;\; u = F_\infty \hat{x}.$$

stabilizes P *and ensures* $\|P \star K\|_\infty < \gamma$.

Proof. If interconnecting $\hat{P}$ from (7.61) with this controller we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \\ \hline v \end{pmatrix} = \left(\begin{array}{cc|c} \hat{A} & B_2 F_\infty & B_1 \\ -\hat{L}_\infty C_2 & \hat{A} + \hat{L}_\infty C_2 + B_2 F_\infty & -\hat{L}_\infty D_{21} \\ \hline -F_\infty & F_\infty & 0 \end{array}\right) \begin{pmatrix} x \\ \hat{x} \\ d \end{pmatrix}.$$

The standard transformation into the error dynamics leads to

$$\left[\begin{array}{cc|c} \hat{A} + B_2 F_\infty & B_2 F_\infty & B_1 \\ 0 & \hat{A} + \hat{L}_\infty C_2 & -(B_1 + \hat{L}_\infty D_{21}) \\ \hline 0 & F_\infty & 0 \end{array}\right] = \left[\begin{array}{c|c} \hat{A} + \hat{L}_\infty C_2 & -(B_1 + \hat{L}_\infty D_{21}) \\ \hline F_\infty & 0 \end{array}\right].$$

Now set $X := \hat{X}_\infty$. Then Lemma 7.36 implies

$$\hat{A}^T X + X\hat{A} + X B_1 B_1^T X + \gamma^{-2} F_\infty^T F_\infty - C_2^T C_2 = 0$$
$$\operatorname{eig}(\hat{A} + B_1 B_1^T X) \subset \mathbb{C}^+.$$

The last statement follows as usual from

$$\hat{A} + B_1 B_1^T X = X^{-1}(\hat{A}^T + (\gamma^{-2} F_\infty^T F_\infty - C_2^T C_2)\hat{X}_\infty)X$$

since $\hat{A} + \hat{X}_\infty(\gamma^{-2} F_\infty^T F_\infty - C_2^T C_2)$ is Hurwitz. Moreover observe that $(\hat{A}, C_2)$ is detectable, $B_1 D_{21}^T = 0$ and $D_{21} D_{21}^T = I$. Theorem 7.10 implies

$$\operatorname{eig}(\hat{A} + \hat{L}_\infty C_2) \subset \mathbb{C}^- \text{ and } \|F_\infty(sI - \hat{A} - \hat{L}_\infty C_2)^{-1}(B_1 + \hat{L}_\infty D_{21})\|_\infty < \gamma.$$

Now Lemma 7.35 finishes the proof.

■

7.11.2 Controller Parameterization

Let us go further and parameterize all suboptimal H_∞-controllers for P. The key technical idea is to dualize and re-describe $\hat{P}$ as done above for P. Let us first transpose $\hat{P}$ to obtain

$$\tilde{P} : \begin{pmatrix} \dot{\tilde{x}} \\ \hline \tilde{z} \\ \tilde{y} \end{pmatrix} = \left(\begin{array}{c|cc} \hat{A}^T & -F_\infty^T & C_2^T \\ \hline B_1^T & 0 & D_{21}^T \\ B_2^T & I & 0 \end{array}\right) \begin{pmatrix} \tilde{x} \\ \hline \tilde{w} \\ \tilde{u} \end{pmatrix}$$

Next, with $\hat{X}_\infty$ we introduce the new variables

$$\tilde{d} = \tilde{w} + \gamma^{-2} F_\infty \hat{X}_\infty \tilde{x} \text{ and } \tilde{v} = \tilde{u} - \hat{L}_\infty^T \tilde{x},\ \hat{L}_\infty = -\hat{X}_\infty C_2^T$$

to arrive at the system

$$\tilde{G}: \begin{pmatrix} \dot{x} \\ \hline \tilde{z} \\ \gamma \tilde{d} \end{pmatrix} = \left(\begin{array}{c|cc} \hat{A}^T + C_2^T \hat{L}_\infty^T & -\gamma^{-1} F_\infty^T & C_2^T \\ \hline B_1^T + D_{21}^T \hat{L}_\infty^T & 0 & D_{21}^T \\ \gamma^{-1} F_\infty \hat{X}_\infty & I & 0 \end{array}\right) \begin{pmatrix} \tilde{x} \\ \hline \gamma \tilde{w} \\ \tilde{v} \end{pmatrix}.$$

Due to the ARE for $\hat{X}_\infty$ also $\tilde{G}$ satisfies the properties in Lemma 7.32. Moreover $\tilde{P}$ is the feedback interconnection of $\tilde{G}$ with

$$\check{P}: \begin{pmatrix} \dot{\tilde{x}} \\ \hline \tilde{v} \\ \tilde{y} \end{pmatrix} = \left(\begin{array}{c|cc} \hat{A}^T - \gamma^{-2} F_\infty^T F_\infty \hat{X}_\infty & -F_\infty^T & C_2^T \\ \hline \hat{L}_\infty^T & 0 & I \\ B_2^T + \gamma^{-2} F_\infty \hat{X}_\infty & I & 0 \end{array}\right) \begin{pmatrix} \tilde{x} \\ \hline \tilde{d} \\ \tilde{d} \end{pmatrix}.$$

Let us "transpose back" which results in

$$P_{\text{tmp}} = \left[\begin{array}{c|cc} \hat{A} - \gamma^{-2} F_\infty^T F_\infty & -\hat{L}_\infty & B_2 + \gamma^{-2} \hat{X}_\infty F_\infty^T \\ \hline -F_\infty & 0 & I \\ C_2 & I & 0 \end{array}\right].$$

Just using the fact that the H_∞-norm of a transfer matrix stays invariant under transposition, we obtain the following result.

Lemma 7.38 *Suppose that Hypothesis 7.33 holds and that there exists some $\hat{X}_\infty \geq 0$ with*

$$\begin{aligned} \hat{A}\hat{X}_\infty + \hat{X}_\infty \hat{A}^T + \hat{X}_\infty(\gamma^{-2} F_\infty^T F_\infty - C_2^T C_2)\hat{X}_\infty + B_1 B_1^T = 0 \\ \operatorname{eig}(\hat{A} + \hat{X}_\infty(\gamma^{-2} F_\infty^T F_\infty - C_2^T C_2)) \subset \mathbb{C}^-. \end{aligned}$$

for $\hat{A} := A + \gamma^{-2} B_1 B_1^T Y_\infty$ and $F_\infty := -B_2^T Y_\infty$.
Then the LTI controller K renders $\hat{P} \star K$ stable and achieves $\|\hat{P} \star K\|_\infty < \gamma$ if and only if $P_{\text{tmp}} \star K \in RH_\infty$ with $\|P_{\text{tmp}} \star K\|_\infty < \gamma$.

Proof. First we need to verify some technical properties.
Since $\hat{A} + B_2 F_\infty$ is Hurwitz, the pair $(\hat{A}, B_2)$ is stabilizable. Set as before $\hat{L}_\infty := -\hat{X}_\infty C_2^T$. Then $(\hat{A}, C_2)$ is detectable, since $\hat{A} + \hat{L}_\infty C_2$ is Hurwitz by Theorem 7.10. Hence $\hat{P}$ is a generalized plant.

Since $B_1 D_{21}^+ = B_1 D_{21}^T (D_{21} D_{21}^T)^{-1} = 0$, $(\hat{A} - B_1 D_{21}^+ C_2, B_1 - B_1 D_{21}^+ D_{21}) = (\hat{A}, B_1)$ is stabilizable by (OF4). Moreover $(\hat{A} + B_2 F_\infty, 0)$ is clealy detectable. Hence $\hat{P}$ satisfies all hypothesis in Lemma 7.28.

The realization of $\tilde{P}$ is the transpose of that of $\hat{P}$. Hence $\tilde{P}$ is also a generalized plant and its realization satisfies the hypothesis in Lemma 7.28.

Next set $H := (D_{21}^T)^+(B_1^T + D_{21}^T\hat{L}_\infty^T)$. Then $H = (D_{21}D_{21}^T)^{-1}D_{21}(B_1^T + D_{21}^T\hat{L}_\infty^T) = \hat{L}_\infty^T$, by (OF2). Moreover $(\hat{A}^T + C_2^T\hat{L}_\infty^T - C_2^T H, (B_1^T + D_{21}^T\hat{L}_\infty^T) - D_{21}^T H) = (\hat{A}^T, B_1^T)$ is detectable and $(\hat{A}^T + C_2^T\hat{L}_\infty^T - \gamma^{-2}F_\infty^T F_\infty \hat{X}_\infty, 0)$ is stabilizable, since $\mathrm{eig}(\hat{A} + \hat{X}_\infty(\gamma^{-2}F_\infty^T F_\infty - C_2^T C_2)) \subset \mathbb{C}^-$. Further since $\hat{A}^T + C_2^T\hat{L}_\infty^T$ is Hurwitz, $\tilde{G}$ is a generalized plant. Hence the realization of $\tilde{G}$ satisfies the hypothesis in Lemma 7.28.

Next we show that $\tilde{G}$ satisfies the hypothesis of Lemma 7.32. We can use Theorem 7.31 to show that $\tilde{G}$ is inner since $\mathrm{eig}(\hat{A}^T + C_2^T\hat{L}_\infty^T) \subset \mathbb{C}^-$.
First observe that

$$\tilde{G}_{21}^{-1} = \left[\begin{array}{c|c} \hat{A}^T + C_2^T\hat{L}_\infty^T - \gamma^{-2}F_\infty^T F_\infty \hat{X}_\infty & \gamma^{-1}F_\infty^T \\ \hline \gamma^{-1}F_\infty \hat{X}_\infty & I \end{array}\right] \in RH_\infty$$

since $\mathrm{eig}(\hat{A} + \hat{X}_\infty(\gamma^{-2}F_\infty^T F_\infty - C_2^T C_2)) \subset \mathbb{C}^-$ by assumption.
Second, due to (OF2), $\hat{L}_\infty = -\hat{X}_\infty C_2^T$ and the ARE for $\hat{X}_\infty$ we have

$$\begin{aligned}
&(\hat{A}^T + C_2^T\hat{L}_\infty^T)^T\hat{X}_\infty + \hat{X}_\infty(\hat{A}^T + C_2^T\hat{L}_\infty) + \begin{pmatrix} B_1^T + D_{21}^T\hat{L}_\infty^T \\ \gamma^{-1}F_\infty\hat{X}_\infty \end{pmatrix}^T \begin{pmatrix} B_1^T + D_{21}^T\hat{L}_\infty^T \\ \gamma^{-1}F_\infty\hat{X}_\infty \end{pmatrix} \\
=&(\hat{A}^T + C_2^T\hat{L}_\infty^T)^T\hat{X}_\infty + \hat{X}_\infty(\hat{A}^T + C_2^T\hat{L}_\infty) + B_1B_1^T + \hat{L}_\infty\hat{L}_\infty^T + \gamma^{-2}\hat{X}_\infty F_\infty^T F_\infty \hat{X}_\infty \\
=&\hat{A}\hat{X}_\infty + \hat{X}_\infty\hat{A}^T - 2\hat{X}_\infty C_2^T C_2\hat{X}_\infty + B_1B_1^T + \hat{L}_\infty\hat{L}_\infty^T + \gamma^{-2}\hat{X}_\infty F_\infty^T F_\infty \hat{X}_\infty \\
=&0.
\end{aligned}$$

This means that $\hat{X}_\infty$ is the observability Gramian for $\tilde{G}$.
Third, we trivially have

$$\begin{pmatrix} -\gamma^{-1}F_\infty \\ C_2 \end{pmatrix}\hat{X}_\infty + \begin{pmatrix} 0 & I \\ D_{21} & 0 \end{pmatrix}\begin{pmatrix} B_1^T + D_{21}^T\hat{L}_\infty^T \\ \gamma^{-1}F_\infty\hat{X}_\infty \end{pmatrix} = 0$$

and

$$\begin{pmatrix} 0 & D_{21}^T \\ I & 0 \end{pmatrix}^T \begin{pmatrix} 0 & D_{21}^T \\ I & 0 \end{pmatrix} = I.$$

Hence $\tilde{G}$ is inner.

For transfer matrices P and K with proper $(I - P_{22}K)^{-1}$, observe that

$$\begin{aligned}
(P \star K)^T &= P_{11}^T + P_{21}^T(I - K^T P_{22}^T)^{-1}K^T P_{12}^T \\
&= P_{11}^T + P_{21}^T K^T(I - P_{22}^T K^T)^{-1}P_{12}^T = P^T \star K^T.
\end{aligned}$$

Therefore $P \star K \in RH_\infty$ with $\|P \star K\|_\infty < 1$ iff $P^T \star K^T \in RH_\infty$ with $\|P^T \star K^T\|_\infty < 1$

This allows us to argue as follows: $\hat{P} \star K \in RH_\infty$ with $\|\hat{P} \star K\|_\infty < \gamma$ iff $\tilde{P} \star K^T \in RH_\infty$ with $\|\tilde{P} \star K\|_\infty < \gamma$ iff (Lemma 7.28 for $\tilde{P}$) K^T stabilizes $\tilde{P}$ with $\|\tilde{P} \star K^T\|_\infty < \gamma$ iff (Lemma 7.35 for $\tilde{P}$, $\tilde{G}$ and $\check{P}$) $\check{P} \star K^T \in RH_\infty$ with $\|\check{P} \star K\|_\infty < \gamma$ iff (transposing) $P_{\text{tmp}} \star K \in RH_\infty$ with $\|P_{\text{tmp}} \star K\|_\infty < \gamma$.

■

The system P_{tmp} has a very particular structure, which allows us to solve $Q = P_{\text{tmp}} \star K$ for K as $K = Q \star P_{\text{tmp}}^{-1}$ and vice versa.

Together with Lemma 7.38, this has the following implication

$$K \text{ stabilizes } P \text{ with } \|P \star K\|_\infty < \gamma$$
$$\Leftrightarrow$$
$$P_{\text{tmp}} \star K \in RH_\infty \text{ with } \|P_{\text{tmp}} \star K\|_\infty < \gamma$$
$$\Leftrightarrow$$
$$K = Q \star P_{\text{tmp}}^{-1} \text{ for some } Q \in RH_\infty \text{ with } \|Q\|_\infty < \gamma.$$

Remark 7.39 All γ-suboptimal H_∞-controllers for P can be expressed as an LFT of a fixed system P_{tmp}^{-1} with a parameter Q that is only constrained to be stable and bounded in H_∞-norm by γ. This means that Q varies in the open RH_∞-ball of radius γ.

Theorem 7.40 *Under the assumptions (OF1)- (OF4), let there exist an LTI controller K which stabilizes P with $\|P \star K\|_\infty < \gamma$. With X_∞ and Y_∞ as in Theorem 7.13 define*

$$Z_\infty := (I - \gamma^{-2} X_\infty Y_\infty)^{-1} \text{ and } F_\infty := -B_2^T Y_\infty,\ L_\infty := -X_\infty C_2^T$$

as well as

$$K_{\text{par}} := \left[\begin{array}{c|cc} A + \gamma^{-2} B_1 B_1^T Y_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$

Then K stabilizes P with $\|P \star K\|_\infty < \gamma$ if and only if $K = K_{\text{par}} \star Q$ with the parameter $Q \in RH_\infty$ satisfying $\|Q\|_\infty < \gamma$.

This explains the terminology for the central controller in Theorem 7.13:

$$K_{\text{par}} \star 0 = \left[\begin{array}{c|c} A + \gamma^{-2} B_1 B_1^T Y_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right].$$

Proof. The system P_{tmp} has a very particular structure: It has a proper inverse since $P_{\text{tmp}}(\infty)$ is invertible and the $(2,2)$-block of P_{tmp} and the $(1,1)$-block of P_{tmp}^{-1} are strictly proper. Hence for LTI systems K and Q, the LFT's $P_{\rm tmp}\star K$ and $Q\star P_{\rm tmp}^{-1}$ are well-posed and LTI.

Moreover the $(1,2)$ and the $(2,1)$ blocks of $P_{\rm tmp}$ are square and invertible. By computation one verifies that $Q=P_{\rm tmp}\star K$ implies $K=Q\star P_{\rm tmp}^{-1}$. By symmetry, $K=Q\star P_{\rm tmp}^{-1}$ implies $Q=P_{\rm tmp}\star K$.

To prove the first implication, fix $\lambda\in\mathbb{C}$ such that it is not a pole of $P_{\rm tmp}$, $P_{\rm tmp}^{-1}$, $(P_{{\rm tmp},21})^{-1}$, K, Q and any complex vector y. Define $u:=K(\lambda)y$ and $z:=Q(\lambda)w$ with $w:=P_{{\rm tmp},21}(\lambda)^{-1}(I-P_{{\rm tmp}_21}(\lambda)K(\lambda))y$. Hence $y=P_{{\rm tmp},21}(\lambda)w+P_{{\rm tmp},22}(\lambda)u$ and $z=P_{{\rm tmp},11}(\lambda)w+P_{{\rm tmp},12}(\lambda)u$. These satisfy

$$\begin{pmatrix} z\\ y\end{pmatrix}=P_{\rm tmp}(\lambda)\begin{pmatrix} w\\ u\end{pmatrix}\Leftrightarrow\begin{pmatrix} w\\ u\end{pmatrix}=P_{\rm tmp}(\lambda)^{-1}\begin{pmatrix} z\\ y\end{pmatrix}.$$

Hence $K(\lambda)y=u=[Q(\lambda)\star P_{\rm tmp}(\lambda)^{-1}]y$, i.e. $K(\lambda)=Q(\lambda)\star P_{\rm tmp}(\lambda)^{-1}$ since y is free. Thus $K=Q\star P_{\rm tmp}^{-1}$ almost everywhere.

It remains to verify the formula for $K_{\rm par}$. Note that $B_2+\gamma^{-2}\hat X_\infty F_\infty^T=(I+\gamma^{-2}\hat X_\infty Y_\infty)B_2$ and

$$I+\gamma^{-2}\hat X_\infty Y_\infty=\hat X_\infty(\hat X_\infty^{-1}+\gamma^{-2}Y_\infty)=\hat X_\infty X_\infty^{-1}=Z_\infty.$$

We hence clearly have

$$P_{\rm tmp}^{-1}=\left[\begin{array}{c|cc}\hat A+B_2F_\infty+\hat L_\infty C & Z_\infty B_2 & -\hat L_\infty\\ \hline -C_2 & 0 & I\\ F_\infty & I & 0\end{array}\right].$$

Finally, the upper $Q\star P_{\rm tmp}^{-1}$ is easily translated into the lower LFT as in

$$Q\star P_{\rm tmp}^{-1}=\left[\begin{array}{c|cc}\hat A+B_2F_\infty+\hat L_\infty C_2 & -\hat L_\infty & Z_\infty B_2\\ \hline F_\infty & 0 & I\\ -C_2 & I & 0\end{array}\right]\star Q.$$

This completes the proof.

■

Exercises

1) Consider the standard ARE $A^TX+XA+XRX+Q=0$ for real matrices A, $R=R^T$ and $Q=Q^T$ and possibly non-Hermitian complex unknowns X.

   If X is a stabilizing solution of the ARE, show that $X=X^T$, $X=\overline{X}$ and that X is unique.

2) Prove Lemma 7.28 and show that

a) If D_{12} has full column rank and $D_{12}^{+} := (D_{12}^T D_{12})^{-1} D_{12}^T$ then (7.52) holds iff

$$(A - B_2 D_{12}^{+} C_1, C_1 - D_{12} D_{12}^{+} C_1) \text{ is detectable.}$$

b) If D_{21} has full row rank and $D_{21}^{+} := D_{21}^T (D_{21} D_{21}^T)^{-1} D_{12}^T$ then (7.53) holds iff

$$(A - B_1 D_{21}^{+} C_2, B_1 - B_1 D_{21}^{+} D_{21}) \text{ is stabilizable.}$$

3) Let $(A, B) \in \mathbb{R}^{n \times (n+m)}$ be stabilizable. With $Q = Q^T$, $R = R^T \geq 0$ define

$$q(x, u) := \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

For $\xi \in \mathbb{R}^n$ let $U(\xi)$ be the set of all pieveweise continuous (p.c.) control functions $u \in L_2[0,\infty)^m$ such that the trajectory of $\dot{x} = Ax + Bu$, $x(0) = \xi$ satisfies $x \in L_2[0,\infty)^n$. Consider the so-called indefinite LQ-problem with stability:

$$v(\xi) = \inf\{\int_0^\infty q(x(t), u(t))dt : \ \dot{x}(t) = Ax(t) + Bu(t), \ x(0) = \xi, \ u \in U(\xi)\}.$$

Suppose that $v(\xi) > -\infty$ for all $\xi \in \mathbb{R}^n$.

a) Show that $v(0) = 0$.

b) Show that v satisfies the dissipation inequality

$$v(x(t_1)) \leq \int_{t_1}^{t_2} q(x(t), u(t))dt + v(x(t_2)) \text{ for any trajectory and } 0 \leq t_1 \leq t_2.$$

c) For $0 \leq t_1 \leq t_2$ show that v satisfies the optimality principle

$$v(x(t_1)) = \inf_{u \text{ p.c. on } [t_1,t_2]} \left(\int_{t_1}^{t_2} q(x(t), u(t))dt + v(x(t_2)) \right).$$

d) Show that v satisfies the parallelogramm identity and that there exists $c > 0$ with $|v(\xi)| \leq c\|\xi\|^2$ for all $\xi \in \mathbb{R}^n$. Argue (without details) why there hence exists a symmetric matrix K with $v(\xi) = \xi^T K \xi$.

4) Let the hypothesis of Exercise 3 hold.

a) Show that K satisfies the linear matrix inequality (LMI)

$$\begin{pmatrix} A^T K + KA & KB \\ B^T K & 0 \end{pmatrix} + \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \geq 0. \tag{7.65}$$

b) Show that $X \leq K$ if $X = X^T$ is any solution of the LMI.

5) In addition to the conditions in Exercise 3 assume that $R > 0$.

a) Show that the cost of the original problem and the one with the data

$$\begin{pmatrix} \tilde{Q} & 0 \\ 0 & I \end{pmatrix} := \begin{pmatrix} I & -SR^{-1} \\ 0 & R^{-1/2} \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I & 0 \\ -R^{-1}S^T & R^{-1/2} \end{pmatrix}$$

and

$$(\tilde{A}, \tilde{B}) := (A, B) \begin{pmatrix} I & 0 \\ -R^{-1}S^T & R^{-1/2} \end{pmatrix}$$

are identical. How are close-to-optimal control functions related? Hence we can assume w.l.o.g. that $S = 0$ and $R = I$, which is done from now on.

b) Show that X satisfies (7.65) if and only if it satisfies the ARI

$$A^T X + XA - XBB^T X + Q \geq 0. \tag{7.66}$$

c) For some C and $T > 0$ let P satisfy the Riccati differential equation $\dot{P} + A^T P + PA - PBB^T P + C^T C = 0$ with $P(T) = 0$ on $[0, T]$. Show that

$$\min_{\dot{x}=Ax+Bu,\ x(0)=\xi, u \text{ p.c.}} \int_0^T \|Cx(t)\|^2 + \|u(t)\|^2 dt = \xi^T P(0) \xi.$$

d) If the minimum in c) is zero for all $\xi \in \mathbb{R}^n$, show that $C = 0$.

e) Show that K even satisfies the ARE that corresponds to (7.66).
HINT: Choose C with $C^T C = A^T K + KA - KBB^T K + Q$ and exploit c), d) and Exercise 3.

6) Consider the ARI $A^T X + XA - XBB^T X + Q \geq 0$ and the corresponding ARE for stabilizable (A, B) and $Q = Q^T$. All unknowns are real-symmetric. An ARE solution X_s is called strong if it satisfies $\operatorname{eig}(A - BB^T X) \subset \mathbb{C}^- \cup \mathbb{C}^0$.

a) If X_s is a strong solution, show that

$$A^T X + XA - XBB^T X + Q \geq 0 \Rightarrow X \leq X_s. \tag{7.67}$$

HINT: Use the difference trick; Prove and work with the fact that $\ker(\Delta)$ for $\Delta = X - X_s$ is $(A - BB^T X_s)$-invariant; continue to argue in coordinates such that $\Delta = \operatorname{diag}(0, \Delta_2)$ with invertible Δ_2.

b) Conversely, if the ARE solution X_s satisfies (7.67), show that X_s is strong. HINT: Use the difference trick and with a modal decomposition of $A - BB^T X_s$ to construct a solution of the ARE that contradicts (7.67).

c) Show that X_s is unique.

d) If the ARI has a solution, show that the ARE has a strong solution.

e) If $Q \geq 0$ show that the ARE has a strong solution and that it is positive semi-definite.

7) Consider $\dot{x} = Ax + B_1 w + B_2 u$, $z = C_1 x + D_{12} u$. Suppose that (A, B_2) is stabilizable, that (A, C_1) has no unobservable modes in $\mathbb{C}^0$ and that

$$D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}.$$

Show that there exists some F with $\mathrm{eig}(A + B_2 F) \subset \mathbb{C}^-$ and $\|(C_1 + D_{12}F)(sI - A - B_2 F)^{-1} B_1\|_\infty < \gamma$ if and only if there exists $X \geq 0$ with

$$A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X + C_1^T C_1 = 0,$$
$$\mathrm{eig}(A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X) \subset \mathbb{C}^-.$$

For "only if" display the $\mathbb{C}^-$-unobservable modes of (A, C_1) in a normal form.

8) Consider $\dot{x} = Ax + B_1 w + B_2 u$, $z = C_1 x + D_{12} u$. Suppose that (A, B_2) is stabilizable and that D_{12} and

$$\begin{pmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{pmatrix} \text{ have full column rank for all } \lambda \in \mathbb{C}. \tag{7.68}$$

Show that there exists some F with $\mathrm{eig}(A + B_2 F) \subset \mathbb{C}^-$ and $\|(C_1 + D_{12}F)(sI - A - B_2 F)^{-1} B_1\|_\infty < \gamma$ if and only if the ARE

$$A^T X + XA + \gamma^{-2} X B_1 B_1^T X - (XB_2 + C_1^T D_{12})(D_{12}^T D_{12})^{-1}(B_2^T X + D_{12}^T C_1) + C_1^T C_1 = 0$$

has a solution $X \geq 0$ with

$$\mathrm{eig}(A + \gamma^{-2} B_1 B_1^T X - B_2 (D_{12}^T D_{12})^{-1}(B_2^T X + D_{12}^T C_1)) \subset \mathbb{C}^-.$$

Can (7.68) be relaxed to requiring the rank condition on $\mathbb{C}^0$ only?

9) View $Y_+(\gamma)$ in Theorem 7.9 as a function of $\gamma > 0$ on $(\gamma_{\min}, \infty)$, where $\gamma_{\min}$ is the infimal $\gamma > 0$ such that the anti-stabilizing solution of the ARE exists. Let γ_{opt} be the optimal value of the state-feedback H_∞-synthesis problem and let $B_1 \neq 0$.

   a) Show that $\gamma_{\min} = \|C_1(sI + A + C_1^T C_1 Y)^{-1} B_1\|_\infty > 0$ with $Y := Y_+(\infty)$.

   b) Show that $Y_+(\cdot)$ is non-decreasing and smooth on its domain of definition.

   c) If $Y_+(\gamma_{\min}) > 0$ show that the optimum is attained.

   d) If $Y_+(\gamma_{\min}) \not> 0$ show that $Y_+(\gamma_{\mathrm{opt}}) \geq 0$ and $Y_+(\gamma_{\mathrm{opt}}) \not> 0$.

   e) Under the latter hypothesis, any infimal sequence F_ν for the optimal state-feedback H_∞-problem satisfies $\|F_\nu\| \to \infty$ (i.e. is high-gain).
   Hint: If not, assume (after taking a subsequence) that F_ν converges to some F_*. Show that $\mathrm{eig}(A + B_2 F_*) \subset \mathbb{C}^- \cup \mathbb{C}^0$, that $(sI - A - B_2 F_*)^{-1} B_1$ is stable and that $\|(C + D_{12}F_*)(sI - A - B_2 F_*)^{-1} B_1\|_\infty = \gamma_{\mathrm{opt}}$. Modify F_* to stabilize $A + B_2 F_*$ without changing the norm.

10) Let P be a generalized plant with the usual state-space realization satisfy (OF1), (OF3) with D_{12}, D_{21}^T of full column rank. Set $D_c = (D_{12}^T D_{12})^{-1} D_{12}^T$ and $D_r = D_{21}^T (D_{21} D_{21}^T)^{-1}$.

a) Show that there exists some K stabilizing P with $\|P \star K\|_\infty < 1$ if and only if there exist symmetric X and Y with $X > 0$, $Y - X^{-1} > 0$ and

$$(A - B_2 D_c C_1)Y + Y(A - B_2 D_c C_1)^T + Y C_1^T (I - D_{12} D_c) C_1 Y + B_1 B_1^T - \\ - B_2 (D_{12}^T D_{12})^{-1} B_2^T < 0$$

$$(A - B_1 D_r C_2)^T X + X(A - B_1 D_r C_2) + X B_1^T (I - D_r D_{21}) B_1 X + C_1 C_1^T - \\ - C_2 (D_{21} D_{21}^T)^{-1} C_2^T < 0.$$

b) Let A be Hurwitz and $Q = Q^T$. For any $\alpha \in \mathbb{R}$ show that $A^T X + XA + Q < 0$ has a solution $X = X^T$ with $X > \alpha I$.

c) Let D_{21} be square and $A - B_1 D_{21}^{-1} C_2$ be Hurwitz. Show that there exists some K stabilizing P with $\|P \star K\|_\infty < 1$ if and only if there exists some $Y > 0$ with

$$(A - B_2 D_c C_1)Y + Y(A - B_2 D_c C_1)^T + Y C_1^T (I - D_{12} D_c) C_1 Y + B_1 B_1^T - \\ - B_2 (D_{12}^T D_{12})^{-1} B_2^T < 0$$

11) Let the realization of the SISO stable $g = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & 1 \end{array}\right]$, $h = \left[\begin{array}{c|c} A_h & B_h \\ \hline C_h & 0 \end{array}\right]$ be minimal and suppose g has only zeros in $\mathbb{C}^+$ that are simple; they are denoted as $z_1, \ldots, z_k$. For $\gamma > 0$ consider the following model-matching problem:

$$\text{There exists } f \in RH_\infty \text{ with } \|h - gf\|_\infty < 1. \tag{7.69}$$

Provide a solution along the following lines:

a) Set up a generalized plant to transform this problem to a standard H_∞-problem and give a solution with one Lyapunov inequality.

b) If X satisfies $(A_g - B_g C_g)X - XA_h - B_g C_h = 0$, show that (7.69) holds if and only if the solution of the following Lyapunov equation is positive definite:

$$(A_g - B_g C_g)Y + Y(A_g - B_g C_g)^T = B_g B_g^T - X B_h B_h^T X^T.$$

HINT: You can use X to transfer the "A"-matrix of the Lyapunov inequality into block-diagonal form. Then one goes from the Lyapunov inequality to the equation.

c) Show that w.l.o.g. it is possible to assume $A_g - B_g C_g = \operatorname{diag}(z_1, \ldots, z_k)$ and that B_g is the all-one vector.

d) With the standard unit vector e_j show that $e_j^T X B_h = h(z_j)$. Derive a formula for $e_j^T Y e_k$.

e) Show that (7.69) holds iff the matrix with entries

$$\frac{1}{z_j + \overline{z_k}} - \frac{h(z_j)\overline{h(z_k)}}{z_j + \overline{z_k}}$$

is positive definite.

8 Robust Performance Synthesis

In the last Section we have dealt with designing a controller that achieves nominal performance. The related problem of minimizing the H_∞-norm of the controlled system has been the subject of intensive research in the 1980's which culminated in the very elegant solution of this problem in terms of Riccati equations.

In view of our analysis results in Section 6, the design of controllers that achieve robust stability and robust performance amounts to minimizing the SSV (with respect to a specific structure) of the controlled system over all frequencies. Although this problem has received considerable attention in the literature, the related optimization problem has, until today, not found any satisfactory algorithmic solution.

Instead, a rather heuristic method has been suggested how to attack the robust performance design problem which carries the name D/K-iteration or scalings/controller-iteration. Although it cannot be theoretically justified that this technique does lead to locally or even globally optimal controllers, it has turned out pretty successful in some practical applications. This is reason enough to describe in this section the pretty simple ideas behind this approach.

8.1 Problem Formulation

We assume that we have built the same set-up us for testing robust performance in Section 6: After fixing the performance signals, pulling out the uncertainties, and including all required uncertainty and performance weightings, one arrives at the controlled uncertain system as described by

$$\begin{pmatrix} z_\Delta \\ z \\ y \end{pmatrix} = P \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} w_\Delta \\ w \\ u \end{pmatrix}, \ u = Ky, \ w_\Delta = \Delta z_\Delta, \ \Delta \in \mathbf{\Delta}$$

under the Hypotheses 6.1. Let us again use the notation $P_\Delta := S(\Delta, P)$. The goal in this section is to design a controller K which stabilizes P_Δ and which leads to

$$\|S(P_\Delta, K)\| \leq 1$$

for all uncertainties $\Delta \in \mathbf{\Delta}$.

In order to formulate again the robust performance analysis test, let us recall the **extended** block-structure

$$\mathbf{\Delta_e} := \left\{ \begin{pmatrix} \Delta_c & 0 \\ 0 & \hat{\Delta}_c \end{pmatrix} : \ \Delta_c \in \mathbf{\Delta_c}, \ \hat{\Delta}_c \in \mathbb{C}^{p_2 \times q_2}, \ \|\hat{\Delta}_c\| < 1 \right\}$$

where p_2/q_2 is the number of components of w/z respectively.

Then K achieves robust performance if it stabilizes the nominal system P and if it renders the inequality

$$\mu_{\boldsymbol{\Delta_e}}(S(P,K)(i\omega)) \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\} \tag{8.1}$$

satisfied.

Finding a K which achieves (8.1) cannot be done directly since not even the SSV itself can be computed directly. The main idea is to achieve (8.1) by guaranteeing that a computable upper bound on the SSV is smaller than one for all frequencies. Let us recall that the set of scalings $\boldsymbol{D}$ that corresponds to $\boldsymbol{\Delta_c}$ for computing an upper bound is given by all

$$D = \begin{pmatrix} D_1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & D_{n_r} & & & & & & \\ & & & D_{n_r+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & D_{n_r+n_c} & & & \\ & & & & & & d_1 I & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & d_{n_f} I \end{pmatrix}$$

where D_j are Hermitian positive definite matrices and d_j are real positive numbers. The class of scalings that corresponds to $\boldsymbol{\Delta_e}$ is then defined as

$$\boldsymbol{D_e} := \left\{ \begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} > 0 \mid D \in \boldsymbol{D} \right\}.$$

Remarks. With choosing this class of scalings, we recall that we ignore the fact that the uncertainty structure comprises real blocks. Moreover, the scaling block that corresponds to the full block included for the performance channel in the extension is set (without loss of generality) equal to the identity matrix.

With these class of scalings we have

$$\mu_{\boldsymbol{\Delta_e}}(S(P,K)(i\omega)) \leq \inf_{D \in \boldsymbol{D_e}} \|D^{-1} S(P,K)(i\omega) D\|.$$

We conclude that any stabilizing controller which achieves

$$\inf_{D \in \boldsymbol{D_e}} \|D^{-1} S(P,K)(i\omega) D\| \leq 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\} \tag{8.2}$$

also guarantees the desired inequality (8.1). Hence instead of designing a controller that reduces the SSV directly, we design one that minimizes the upper bound of the SSV which is obtained with the class of scalings $\boldsymbol{D_e}$: SSV design is, actually, upper bound design!

Let us slightly re-formulate (8.2) equivalently as follows: There exists a frequency dependent scaling $D(\omega) \in \boldsymbol{D}_e$ such that

$$\|D(\omega)^{-1} S(P, K)(i\omega) D(\omega)\| < 1 \ \text{ for all } \ \omega \in \mathbb{R} \cup \{\infty\}.$$

This leads us to the precise formulation of the problem that we intend to solve.

Robust performance synthesis problem. Minimize

$$\sup_{\omega \in \mathbb{R} \cup \{\infty\}} \|D(\omega)^{-1} S(P, K)(i\omega) D(\omega)\| \tag{8.3}$$

over all controllers K that stabilize P, and over all frequency dependent scalings $D(\omega)$ with values in the set $\boldsymbol{D_e}$.

If the minimal value that can be achieved is smaller than one, we are done: We guarantee (8.2) and hence also (8.1).

If the minimal value is larger than one, the procedure fails. Since we only consider the upper bound, it might still be possible to push the SSV below one by a suitable controller choice. Hence we cannot draw a definitive conclusion in this case. In practice, one concludes that robust performance cannot be achieved, and one tries to adjust the weightings in order to still be able to push the upper bound on the SSV below one.

8.2 The Scalings/Controller Iteration

Unfortunately, it is still not possible to minimize (8.3) over the controller K and the frequency dependent scalings $D(\omega)$ **together**. Therefore, it has been suggested to iterate the following two steps: 1) Fix the scaling function $D(\omega)$ and minimize (8.3) over all stabilizing controllers. 2) Fix the stabilizing controller K and minimize (8.3) over all scaling functions $D(\omega)$.

This procedure is called D/K-iteration and we use the more appropriate terminology scalings/controller iteration. It does not guarantee that we really reach a local or even a global minimum of (8.3). Nevertheless, in each step of this procedure the value of (8.3) is *reduced*. If it can be rendered smaller than one, we can stop since the desired goal is achieved. Instead, one could proceed until one cannot reduce the value of (8.3) by any of the two steps.

Let us now turn to the details of this iteration.

First Step. Set

$$D_1(\omega) = I$$

and minimize

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|D_1(\omega)^{-1}S(P,K)(i\omega)D_1(\omega)\| = \|S(P,K)\|_\infty$$

over all K that stabilize P. This is nothing but a standard H_∞ problem! Let the optimal value be smaller than γ_1, and let the controller K_1 achieve this bound.

After Step k we have found a scaling function $D_k(\omega)$ and a controller K_k that stabilizes P and which renders

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|D_k(\omega)^{-1}S(P,K_k)(i\omega)D_k(\omega)\| < \gamma_k \tag{8.4}$$

for some bound γ_k satisfied.

Scalings optimization to determine $D_{k+1}(\omega)$. Given K_k, perform an SSV robust performance analysis test. This amounts to calculating at each frequency $\omega \in \mathbb{R}\cup\{\infty\}$ the upper bound

$$\inf_{D\in\boldsymbol{D}_e} \|D^{-1}S(P,K_k)(i\omega)D\| \tag{8.5}$$

on the SSV. Typical algorithms also return an (almost) optimal scaling $D_{k+1}(\omega)$. This step leads to a scaling function $D_{k+1}(\omega)$ such that

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|D_{k+1}(\omega)^{-1}S(P,K_k)(i\omega)D_{k+1}(\omega)\| < \hat{\gamma}_k$$

for some new bound $\hat{\gamma}_k$.

Controller optimization to determine K_{k+1}. We cannot optimize (8.3) over K for an arbitrary scaling function $D(\omega) = D_{k+1}(\omega)$. This is the reason why one first has to fit this scaling function by a real rational $\hat{D}(s)$ that is proper and stable, that has a proper and stable inverse, and that is chosen close to $D(\omega)$ in the following sense: For a (small) error bound $\epsilon > 0$, it satisfies

$$\|D_{k+1}(\omega) - \hat{D}(i\omega)\| \leq \epsilon \ \text{ for all } \ \omega \in \mathbb{R}\cup\{\infty\}.$$

With $\hat{D}$, one then solves the H_∞-control problem

$$\inf_{K \text{ stabilizes } P} \|\hat{D}^{-1}S(P,K)\hat{D}\|_\infty$$

to find an almost optimal controller K_{k+1}. This step leads to a K_{k+1} such that

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|D_{k+1}(\omega)^{-1}S(P,K_k)(i\omega)D_{k+1}(\omega)\| < \gamma_{k+1} \tag{8.6}$$

holds for some new bound γ_{k+1}.

We have arrived at (8.4) for $k \to k+1$ and can iterate.

Let us now analyze the improvements that can be gained during this iteration. During the scalings iteration, we are guaranteed that the new bound $\hat{\gamma}_k$ can be chosen with

$$\hat{\gamma}_k < \gamma_k.$$

However, it might happened that the value of (8.5) cannot be made significantly smaller than γ_k at some frequency. Then the new bound $\hat{\gamma}_k$ is close to γ_k and the algorithm is stopped. In the other case, $\hat{\gamma}_k$ is significantly smaller than γ_k and the algorithm proceeds.

During the controller iteration, one has to perform an approximation of the scaling function $D_{k+1}(\omega)$ by $\hat{D}(i\omega)$ uniformly over all frequencies with a real rational $\hat{D}(s)$. If the approximation error is small, we infer that

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|\hat{D}^{-1}(i\omega)S(P,K)(i\omega)\hat{D}(i\omega)\| \approx \sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|D_{k+1}^{-1}(\omega)S(P,K)(i\omega)D_{k+1}(\omega)\|$$

for both $K = K_k$ and $K = K_{k+1}$. For a sufficiently good approximation, we can hence infer from (8.6) that

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|\hat{D}^{-1}(i\omega)S(P,K_k)(i\omega)\hat{D}(i\omega)\| < \hat{\gamma}_k.$$

Since K_{k+1} is obtained by solving an H_∞-optimization problem, it can be chosen with

$$\|\hat{D}^{-1}S(P,K_{k+1})\hat{D}\|_\infty \leq \|\hat{D}^{-1}S(P,K_k)\hat{D}\|_\infty.$$

This implies

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|\hat{D}^{-1}(i\omega)S(P,K_{k+1})(i\omega)\hat{D}(i\omega)\| < \hat{\gamma}_k.$$

Again, if the approximation of the scalings is good enough, this leads to

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \|D_{k+1}^{-1}(\omega)S(P,K_{k+1})(i\omega)D_{k+1}(\omega)\| < \hat{\gamma}_k.$$

Hence the bound γ_{k+1} can be taken **smaller** than $\hat{\gamma}_k$. Therefore, we can conclude

$$\gamma_{k+1} < \gamma_k.$$

Note that this inequality requires a good approximation of the scalings. In practice it can very well happen that the new bound γ_{k+1} cannot be made smaller than the previous bound γ_k! In any case, if γ_{k+1} can be rendered significantly smaller than γ_k, the algorithm proceeds, and in the other cases it stops.

Under ideal circumstances, we observe that the sequence of bounds γ_k is monotonically decreasing and, since bounded from below by zero, hence convergent. This is the **only convergence conclusion that can be drawn.** There are no general implications about the convergence of K_k or of the scaling functions $D_k(\omega)$, and no conclusions can be drawn about optimality of the limiting value of γ_k.

Remarks.

- The μ-tools support the fitting of $D(\omega)$ with rational transfer matrices $\hat{D}(i\omega)$. This is done with GUI support on an element by element basis of the function $D(\omega)$, where the user has control over the McMillan degree of the rational fitting function. In addition, automatic fitting routines are available as well.
- Since $\hat{D}$ and $\hat{D}^{-1}$ are both proper and stable, minimizing $\|\hat{D}^{-1}S(P,K)\hat{D}\|_\infty$ over K amounts to solving a standard weighted H_∞-problem. The McMillan degree of an (almost) optimal controller is given by

$$2 * \text{McMillan degree of } \hat{D} + \text{McMillan degree of } P.$$

  Keeping the order of K small requires to keep the order of $\hat{D}$ small. Note that this might be only possible at the expense of an large approximation error during the fitting of the scalings. Again, no general rules can be given here and it remains to the user to find a good trade-off between these two aspects.
- If the order of the controller is too large, one should perform a reduction along the lines as described in the chapters 7 and 9 in [ZDG] and chapter 11 in [SP].

To learn about the practice of applying the scalings/controller we recommend to run the `himat`-demo around the design of a pitch axis controller for the simple model of an airplane which is included in the μ-Toolbox. This examples comprises a detailed description of how to apply the commands for the controller/scalings iteration. Moreover, it compares the resulting μ-controller with a simple loop-shaping controller and reveals the benefits of a robust design.

Exercise

1) Consider the following simplified model of a distillation column

$$G(s) = \frac{1}{75s+1}\begin{pmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{pmatrix}$$

(with time expressed in minutes) in the tracking configuration of Figure 61. The uncertainty weighting is given as

$$W(s) = \begin{pmatrix} \frac{s+0.1}{0.5s+1} & 0 \\ 0 & \frac{s+0.1}{0.5s+1} \end{pmatrix}$$

and the real rational proper and stable Δ with $\|\Delta\|_\infty < 1$ is structured as

$$\Delta(s) = \begin{pmatrix} \Delta_1(s) & 0 \\ 0 & \Delta_2(s) \end{pmatrix}.$$

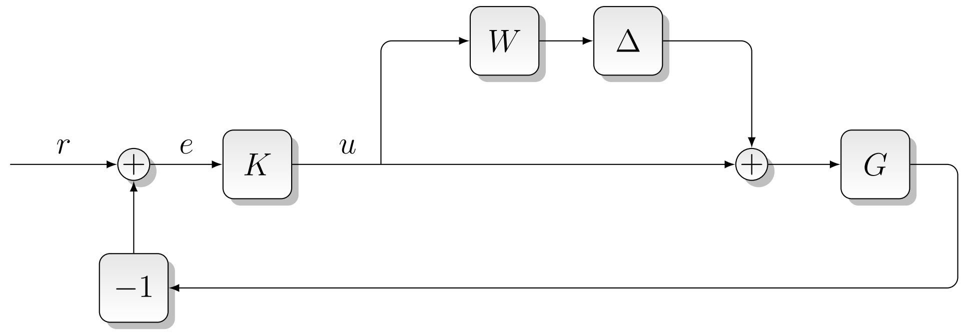


Figure 61: Tracking interconnection of simplified distillation column.

a) Provide an interpretation of this uncertainty structure, and discuss how the uncertainty varies with frequency.

b) Choose the decoupling controllers

$$K_\beta(s) = \frac{\beta}{s} G(s)^{-1}, \quad \beta \in \{0.1, 0.3, 0.5, 0.7\}$$

Discuss in terms of step responses in how far these controller lead to a good closed-loop response! Now choose the performance weightings

$$W_p(s) = \alpha \frac{s+\beta}{s+10^{-6}} I, \quad \alpha \in [0,1], \ \beta \in \{0.1, 0.3, 0.5, 0.7\}$$

with a small perturbation to render the pure intergrator stable. Test robust stability, nominal performance, and robust performance of K_β for $\alpha = 1$ and discuss.

Reveal that K_β does not lead to good robust performance in the time-domain by determining a simple *real* uncertainty which leads to bad step responses in $r \to e$.

c) Perform a two step scalings/controller iteration to design a controller that enforces robust peformance, possibly by varying α to render the SSV for robust performance of the controlled system close to one. Discuss the resulting controller in the frequency-domain and the time-domain, and compare it with the decoupling controllers K_β. Show that this controller exhibits a much better time-response for the 'bad' disturbance constructed in the previous exercise.

Remark. You should make a sensible selection of all the plots that you might collect during performing this exercise in order to nicely display in a non-repetitive fashion all the relevant aspects of your design!

9 Youla Jabr Bongiorno Kucera Parametrizations

Our main goal in this section is to parameterize all LTI controller K that stabilize a given LTI generalized plant P. Theorem 9.16 will show that under some hypothesis K stabilizes P iff there exist some stable Q with $K = J \star Q$. A state-space realization of J is given in Hypothesis 9.15. To develop this result, we will start from a more general setting. More precisely we will replace the set of stable systems RH_∞ by an integral domain R with unit 1 and replace the set of systems $\mathbb{R}(s)$, the real rational functions, by F, the field of fractions corresponding to R. We introduce in Section 9.1 the algebraic framework for such general systems and develop algebraic stabilization theory in Section 9.2. Afterwards the Youla parametrization (Theorem 9.13) allows us to parametrize all stabilizing controllers for general systems and the double Bézout identity for LTI systems in Section 9.3 allows us to give explicit state space realizations of the systems in Theorem 9.13 in the LTI case. This will finally lead to Theorem 9.16.

9.1 Algebraic Framework

Let R be an integral domain with unit 1. An integral domain is a nonzero commutative ring in which the product of any two nonzero elements is nonzero. Moreover let F be the corresponding field of fractions, which is the smallest field in which R can be embedded. Sometimes F is denoted by $\mathrm{Quot}(R)$ or $\mathrm{Frac}(R)$. We denote the set of matrices with entries in R by $M(R)$ or $R^{m\times n}$, if we want to specify the dimension of the matrices. Analogously we denote the set of matrices with entries in F by $M(F)$ or $F^{m\times n}$. The identity matrix is denoted as I. If dimensions are not specified and if multiplying or adding matrices, dimensions are assumed to be compatible. In the sequel, elements in $M(F)$ are called systems and those in $M(R)$ are called stable systems.

Example 9.1

- With $R = RH_\infty$ and $F = \mathbb{R}(s)$, the real-rational functions, the results in this section can be spezialized to what we are familiar with.
- $R = H_\infty$, the Banach algebra of bounded holomorphic functions in $\mathbb{C}_>$. This e.g. allows to handle systems with multiple delays.
- R equals the set of real-rational functions with all its poles in $S \subset \mathbb{C} \cup \{\infty\}$. If $S = \{\infty\}$, then R is the set of all real polynomials which are not constant.
- R equals the set of n/d with $n \in \mathbb{R}[z_1, \ldots, z_m]$ and with $d \in \mathbb{R}[z_1, \ldots, z_m] \setminus \{0\}$ having all zeros in $\mathbb{C}^n \setminus \mathbb{D}^n$ where $\mathbb{D}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| \leq 1\}$. This allows to handle stable multi-dimensional discrete-time systems.

In order to develop the algebraic stabilization theory we need to make use of right and left coprime factorizations of systems.

Definition 9.2 *$G \in M(F)$ has a* **right factorization** *(over R) if there exist $M, N \in M(R)$ such that M is square and invertible in $M(F)$ with $G = NM^{-1}$. The factorization is* **coprime** *if there exist $X, Y \in M(R)$ with $XM + YN = I$.*

Any $G \in F^{m\times n}$ defines a linear map $G : F^n \to F^m$. If G has a right factorization observe that $y = Gu$ iff $y = N\xi$ and $u = M\xi$ for some $\xi \in F^n$. Hence we infer

$$\left\{ \begin{pmatrix} y \\ u \end{pmatrix} : \ y = Gu, \ u \in F^m \right\} = \operatorname{im} \begin{pmatrix} N \\ M \end{pmatrix}.$$

This is a so-called image representation (of the graph) of the system G.

The advantage of stability of the factors emerges if considering "bounded input pairs" $(y, u) \in R^m \times R^n$ with $y = Gu$ only: all pairs $(N\xi, M\xi)$ for $\xi \in R^n$ clearly have this property. Coprimeness implies that we do indeed reach all such pairs in this fashion.

Lemma 9.3 *If $G = NM^{-1}$ is a right coprime factorization (rcf) then*

$$\left\{ \begin{pmatrix} y \\ u \end{pmatrix} : \ u \in R^n, y = Gu \in R^m \right\} = \left\{ \begin{pmatrix} N \\ M \end{pmatrix} \xi : \ \xi \in R^m \right\}.$$

Proof. "$\supset$": If $y = N\xi$ and $u = M\xi$ for $\xi \in R^n$, we infer $y \in R^m$, $u \in R^n$ and $\xi = M^{-1}u$ implying $y = NM^{-1}u$.

"$\subset$": If $y = Gu = NM^{-1}u$ for $y \in R^m$, $u \in R^n$, let $\xi := M^{-1}u$ to get

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} N \\ M \end{pmatrix} \xi.$$

Co-primeness just means that $\begin{pmatrix} N \\ M \end{pmatrix}$ has a left-inverse $L = \begin{pmatrix} X & Y \end{pmatrix} \in M(R)$. We infer

$$R^n \ni L \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} N \\ M \end{pmatrix} \xi = \xi$$

and thus $\xi \in R^n$. (Hence ξ must be "bounded").

■

It is clear how to define the dual notion of Definition 9.2.

Definition 9.4 *$G \in M(F)$ has a* **left factorization** *(over R) if there exist $M, N \in M(R)$ such that M is square and invertible in $M(F)$ with $G = M^{-1}N$. The factorization is* **coprime** *if $\begin{pmatrix} M & N \end{pmatrix}$ has a right-inverse in $M(R)$.*

Left-factorizations give rise to so-called kernel representations:

$$\left\{ \begin{pmatrix} y \\ u \end{pmatrix} : \; y = Gu, \; u \in F^m \right\} = \ker \begin{pmatrix} M & -N \end{pmatrix}.$$

Example 9.5 Let G be a transfer matrix with realization $\dot{x} = Ax + Bu$, $y = Cx + Du$.

1) If F renders $A + BF$ Hurwitz then the pre-compensation $u = Fx + \xi$ leads to a right factorization: Indeed we obtain

$$\begin{aligned} \dot{x} &= (A + BF)x + B\xi \\ y &= (C + DF)x + D\xi \\ u &= Fx + I\xi. \end{aligned}$$

This motivates to define

$$\begin{pmatrix} N \\ M \end{pmatrix} = \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & D \\ F & I \end{array} \right] \in RH_\infty. \tag{9.1}$$

Since I is invertible we get $M^{-1} = \left[\begin{array}{c|c} A & B \\ \hline -F & I \end{array} \right]$ and hence

$$NM^{-1} = \left[\begin{array}{cc|c} A + BF & -BF & B \\ \hline 0 & A & B \\ C + DF & -DF & D \end{array} \right] = \left[\begin{array}{cc|c} A & BF & B \\ \hline 0 & A + BF & 0 \\ C & -DF & D \end{array} \right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = G.$$

The second equation follows by transforming the system with $T = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$.

2) The factorization in 1) is however in general not coprime. To see this let $G = \left[\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \end{array} \right] = 1$. Clearly $F = -1$ renders $A - BF$ Hurwitz and

$$M^{-1} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array} \right] = \frac{s+1}{s} \quad \text{and} \quad N = \left[\begin{array}{c|c} -1 & 1 \\ \hline -1 & 1 \end{array} \right] = \frac{s}{s+1}.$$

Now suppose that the factorization is coprime. Choose $u = 1 \in RH_\infty$ to get $y = Gu = 1 \in RH_\infty$. By Lemma 9.3 there exists $\xi \in RH_\infty$ with $1 = u = N\xi = \frac{s}{s+1}$. Hence $\xi = \frac{s+1}{s} \notin RH_\infty$.

3) If A is Hurwitz the factorization in 1) is coprime since $X = M^{-1} \in RH_\infty$ and $Y = 0 \in RH_\infty$ implies $XM + YN = I$.

4) If L renders $A + LC$ Hurwitz, we get analogously to 1) a left factorization as

$$\begin{pmatrix} \tilde{M} & \tilde{N} \end{pmatrix} = \left[\begin{array}{c|cc} A + LC & L & B + LD \\ \hline C & I & D \end{array}\right].$$

If in addition F renders $A + BF$ Hurwitz, we will see in Section 9.3 that the above left factorization and the right factorization (9.1) are coprime.

Definition 9.6 *$Q \in M(R)$ is called* **unimodular** *or a* **unit** *if Q^{-1} exists and satisfies $Q^{-1} \in M(R)$.*

The next result shows that rcf's only differ by unimodular right factors.

Lemma 9.7 *If $N_1 M_1^{-1} = N_2 M_2^{-1}$ are two right coprime factorizations then there exists a unimodular Q with $M_1 = M_2 Q$ and $N_1 = N_2 Q$.*

Proof. If $N_1 M_1^{-1} = N_2 M_2^{-1}$ are two coprime factorizations, we infer

$$\begin{pmatrix} M_1 \\ N_1 \end{pmatrix} Q = \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \text{ and } \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} Q^{-1} = \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}$$

for $Q := M_1^{-1} M_2$. Left-multiplying with the stable left-inverses L_1 and L_2 corresponding to the two factorizations implies

$$Q = L_1 \begin{pmatrix} M_2 \\ N_2 \end{pmatrix} \in M(R) \text{ and } Q^{-1} = L_2 \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \in M(R).$$

■

Of course, dually, left coprime factorizations just differ by unimodular left factors.

In RH_∞, exactly the proper and stable transfer matrices with a proper and stable inverse are the units. Such transfer matrices are often called **outer** or **minimum-phase**.

In this context you should keep in mind the topological consequences of the small-gain theorem. For example, if $M \in RH_\infty$ and $\|M\|_\infty < 1$ then $I - M$ is a unit.

Lemma 9.8 *Let $A, B \in M(F)$ and let AB be a unit.*

- *If $A \in M(R)$ and B is invertible then B^{-1} is stable.*
- *If $B \in M(R)$ and A is invertible then A^{-1} is stable*

Proof. If $AB = Q$ is a unit then $Q^{-1}A \in M(R)$ is a right inverse of B and since B is invertible we get $Q^{-1}A = B^{-1}$. This shows $B^{-1} \in M(R)$. The second statement follows analogously.

■

9.2 Stabilizable Plants and Controller Parametrization

We now define stabilization of a system G by a controller K in (almost) complete analogy to Section 3.

Definition 9.9 *The system $G \in M(F)$ is stabilized by $K \in M(F)$ if*

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1} \in M(R).$$

Precisely it is meant that the inverse exists in $M(F)$ and is stable.

Let us **assume** that G admits the rcf $G = NM^{-1}$. Then trivially

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1} = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M & -K \\ -N & I \end{pmatrix}^{-1}. \tag{9.2}$$

Let us first see that stabilizing controllers are characterized by stability of the inverse on the right.

Lemma 9.10 *If $G = NM^{-1}$ is a rcf then K stabilizes G iff*

$$\begin{pmatrix} M & -K \\ -N & I \end{pmatrix}^{-1} \in M(R). \tag{9.3}$$

Proof. "If" is trivial by (9.2) since $M \in M(R)$. For "only if", $XM + YN = I$ implies

$$\begin{pmatrix} X & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & -Y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M & -K \\ -N & I \end{pmatrix} = I.$$

Right-multiplying (9.3) shows by (9.2) that the inverse is stable.

■

Theorem 9.11 *$G \in M(F)$ with the rcf NM^{-1} is stabilizable iff there exist $\tilde{X}, \tilde{Y} \in M(R)$ such that $\tilde{X}M - \tilde{Y}N = I$ and $\tilde{X}$ is invertible.*

Proof. To show "only if", let K stabilize G and define

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{N} & \tilde{M} \end{pmatrix} := \begin{pmatrix} M & -K \\ -N & I \end{pmatrix}^{-1} \in M(R). \tag{9.4}$$

This implies

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ 0 & I \end{pmatrix} \begin{pmatrix} M & -K \\ -N & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -N & I \end{pmatrix}. \tag{9.5}$$

Hence $\tilde{X}M - \tilde{Y}N = I$ and $\tilde{X}$ is invertible, since the r.h.s. is invertible. We also infer $K = \tilde{X}^{-1}\tilde{Y}$ and that this is a lcf.

For "if", let $\tilde{X}$, $\tilde{Y}$ be as described. Then $K := \tilde{X}^{-1}\tilde{Y}$ implies (9.5). Since the left factor on the left is stable and the r.h.s. is a unit, Lemma 9.8 implies stability of $\begin{pmatrix} M & -K \\ -N & I \end{pmatrix}$. By Lemma 9.10 this implies that K stabilizes G.

■

The proof shows that stabilizing controllers admit a lcf $\tilde{X}^{-1}\tilde{Y}$ with factors as in Theorem 9.11. Moreover if $\tilde{X}$, $\tilde{Y}$ are taken as in Theorem 9.11 then $K = \tilde{X}^{-1}\tilde{Y}$ stabilizes G. Let's see what happens if G admits both a rcf and a lcf.

Lemma 9.12 *Let G have both the rcf NM^{-1} and the lcf $\tilde{M}^{-1}\tilde{N}$ and $\tilde{X}, \tilde{Y} \in M(R)$ satisfy $\tilde{X}M - \tilde{Y}N = I$. Then*

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M \\ -N \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{N} & \tilde{M} \end{pmatrix} \text{ is a unit.}$$

Proof. The equation is a consequence of $\tilde{M}^{-1}\tilde{N} - NM^{-1} = 0$. If we take stable X, Y with $\tilde{M}X - \tilde{N}Y = I$, we obtain

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & -Y \\ -N & X \end{pmatrix} = \begin{pmatrix} I & \tilde{Y}X - \tilde{X}Y \\ 0 & I \end{pmatrix}.$$

All these matrices are stable and the r.h.s. is obviously even a unit. Since the r.h.s is invertible the same holds for both matrices on the l.h.s. By Lemma 9.8, all matrices are units and the claim follows.

■

Note that since G has the rcf NM^{-1} there clearly always exist $\tilde{X}$ $\tilde{Y} \in M(R)$ with $\tilde{X}M - \tilde{Y}N = I$.

Theorem 9.13 *(Youla parametrization) Let $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be a rcf and lcf of G and choose $\tilde{X}, \tilde{Y} \in M(R)$ with $\tilde{X}M - \tilde{Y}N = I$. Then K stabilizes G iff*

$$K = (\tilde{X} + Q\tilde{N})^{-1}(\tilde{Y} + Q\tilde{M}) \tag{9.6}$$

for some Youla parameter $Q \in M(R)$ such that $\tilde{X} + Q\tilde{N}$ is invertible.

For a controller described as in (9.6), Q enters affinely in

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1} = \begin{pmatrix} M\tilde{X} & M\tilde{Y} \\ N\tilde{X} & N\tilde{Y}+I \end{pmatrix} + \begin{pmatrix} M \\ N \end{pmatrix} Q \begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix}. \tag{9.7}$$

Note that (9.6) is a lcf of K. If K_0 stabilizes $G = NM^{-1}$, we have seen in Theorem 9.11 that it admits a lcf $K_0 = \tilde{X}^{-1}\tilde{Y}$ with stable $\tilde{X}$, $\tilde{Y}$ such that $\tilde{X}M - \tilde{Y}N = I$ and $\tilde{X}$ is invertible. Therefore, the parametrization in Theorem 9.13 can be based on **any** stabilizing controller K_0 for G, and K_0 is then reobtained back from (9.6) with $Q = 0$.

Proof. " $\Leftarrow$ ": left-multiply the equation in Lemma 9.12 with $\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} \in M(R)$ to get

$$\begin{pmatrix} \tilde{X} + Q\tilde{N} & \tilde{Y} + Q\tilde{M} \\ \tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M \\ -N \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Hence (9.6) is a lcf of K. Moreover we trivially infer

$$\begin{pmatrix} \tilde{X} + Q\tilde{N} & \tilde{Y} + Q\tilde{M} \\ 0 & I \end{pmatrix} \begin{pmatrix} M & -K \\ -N & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -N & I \end{pmatrix}.$$

By Lemma 9.8 and 9.10 we conclude that K stabilizes G. Right-multiplying the inverse of the right-factor and using (9.2), the last equation clearly implies

$$\begin{pmatrix} M & 0 \\ N & I \end{pmatrix} \begin{pmatrix} \tilde{X} + Q\tilde{N} & \tilde{Y} + Q\tilde{M} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1}.$$

This is the claimed expression for the inverse.

"$\Rightarrow$": Let $\tilde{K}$ be any stabilizing controller. Lemma 9.12 implies

$$\begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & -\tilde{K} \\ -N & I \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & T \end{pmatrix}$$

for some S and T. Moreover, since both matrices on the left-hand side are invertible the right-hand side is invertible. By left-multiplying the inverse of the r.h.s., we get with

$Q := -ST^{-1}$ that

$$\begin{pmatrix} \tilde{X} + Q\tilde{N} & \tilde{Y} + Q\tilde{M} \\ T^{-1}\tilde{N} & T^{-1}\tilde{M} \end{pmatrix} = \begin{pmatrix} M & -\tilde{K} \\ -N & I \end{pmatrix}^{-1} \in M(R). \tag{9.8}$$

This implies that $\tilde{X} + Q\tilde{N}$ and $\tilde{Y} + Q\tilde{M}$ are stable. As in the proof of Theorem 9.11, we also infer that $\tilde{X} + Q\tilde{N}$ is invertible and that we have $\tilde{K} = (\tilde{X} + Q\tilde{N})^{-1}(\tilde{Y} + Q\tilde{M})$. (9.8) even shows that this is a lcf.

Stability of Q itself follows from Lemma 9.12 and

$$\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{N} & \tilde{M} \end{pmatrix} = \begin{pmatrix} \tilde{X} + Q\tilde{N} & \tilde{Y} + Q\tilde{M} \\ \tilde{N} & \tilde{M} \end{pmatrix} \in M(R).$$

■

9.3 Double Bézout Identity for LTI Systems

If $G \in \mathbb{R}(s)$ is proper, the ingredients to parametrize all stabilizing controllers can be obtained by state-space computations.

For this purpose we choose a stabilizable and detectable realization

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

If F, L are taken with $\text{eig}(A + BF) \subset \mathbb{C}_<$, $\text{eig}(A + LC) \subset \mathbb{C}_<$ then

$$K = \left[\begin{array}{c|c} (A + BF) + L(C + DF) & -L \\ \hline F & 0 \end{array}\right]$$

is an observer-based stabilizing controller for G. It is now not difficult to construct matrices in RH_∞ which satisfy the **double Bézout identity**

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I \tag{9.9}$$

and $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ (as well as $K = \tilde{X}^{-1}\tilde{Y} = YX^{-1}$).

Introduce the abbreviations $A_F := A + BF$, $C_F = C + DF$ as well as $A_L := A + LC$, $B_L := B + LD$. Then

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1} = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & (A+BF)+L(C+DF) & 0 & -L \\ \hline 0 & -F & I & 0 \\ -C & 0 & -D & I \end{array}\right]^{-1} =$$

$$= \left[\begin{array}{cc|cc} A & BF & B & 0 \\ -LC & A+BF+LC & -LD & -L \\ \hline 0 & F & I & 0 \\ C & DF & D & I \end{array}\right] = \left[\begin{array}{cc|cc} A_F & BF & B & 0 \\ 0 & A_L & -B_L & -L \\ \hline F & F & I & 0 \\ C_F & DF & D & I \end{array}\right] =$$

$$= \left[\begin{array}{c|cc} A_F & B & 0 \\ \hline F & I & 0 \\ C_F & D & I \end{array}\right] \left[\begin{array}{c|cc} A_L & -B_L & -L \\ \hline F & I & 0 \\ 0 & 0 & I \end{array}\right] =: \begin{pmatrix} M & 0 \\ N & I \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \\ 0 & I \end{pmatrix}.$$

In complete analogy but with another coordinate change:

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1} = \left[\begin{array}{cc|cc} A & BF & B & 0 \\ -LC & A+BF+LC & -LD & -L \\ \hline 0 & F & I & 0 \\ C & DF & D & I \end{array}\right] =$$

$$= \left[\begin{array}{cc|cc} A_L & 0 & B_L & L \\ -LC & A_F & -LD & -L \\ \hline 0 & F & I & 0 \\ C & C_F & D & I \end{array}\right] \left[\begin{array}{cc|cc} A_F & -LC & -LD & -L \\ 0 & A_L & B_L & L \\ \hline F & 0 & I & 0 \\ C_F & C & D & I \end{array}\right] =$$

$$= \left[\begin{array}{c|cc} A_F & 0 & -L \\ \hline F & I & 0 \\ C_F & 0 & I \end{array}\right] \left[\begin{array}{c|cc} A_L & B_L & L \\ \hline 0 & I & 0 \\ C & D & I \end{array}\right] =: \begin{pmatrix} I & Y \\ 0 & X \end{pmatrix} \begin{pmatrix} I & 0 \\ \tilde{N} & \tilde{M} \end{pmatrix}.$$

Note that, by their respective definitions, M, $\tilde{X}$ and $\tilde{M}$, X have proper inverses.

Hence

$$\begin{pmatrix} M \\ N \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} Y \\ X \end{pmatrix} \begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

which implies

$$\begin{pmatrix} M \\ N \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \end{pmatrix} - \begin{pmatrix} Y \\ X \end{pmatrix} \begin{pmatrix} \tilde{N} & \tilde{M} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and thus

$$\begin{pmatrix} M & Y \\ N & X \end{pmatrix} \begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} = I.$$

We can also get to the equivalent version

$$\begin{pmatrix} M & -Y \\ -N & X \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{N} & \tilde{M} \end{pmatrix} = I.$$

Note again that, by the definition of M, N, $\tilde{N}$, $\tilde{M}$, X, Y, $\tilde{X}$ and $\tilde{Y}$ both matrices on the l.h.s. have proper inverses. Hence (9.9) holds and in particular $\tilde{X}M - \tilde{Y}N = I$. These (or those with reversed order) are called double Bézout identities.

From

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix} \begin{pmatrix} M & 0 \\ N & I \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \\ 0 & I \end{pmatrix} = I$$

we infer $(-GM + N)\tilde{X} = 0$ and thus

$$G = NM^{-1}.$$

In a similar fashion one can extract $K = \tilde{X}^{-1}\tilde{Y}$. The double Bézout identity then implies $\tilde{N}M - \tilde{M}N = 0$ and thus

$$G = \tilde{M}^{-1}\tilde{N}.$$

Hence the assumptions from Theorem 9.13 are satisfied and every controller $K_Q = (\tilde{X} + Q\tilde{N})^{-1}(\tilde{Y} + Q\tilde{M})$ with $Q \in M(R)$ such that $\tilde{X} + Q\tilde{N}$ is invertible, stabilizes G. Moreover we regain the controller K by choosing $Q = 0$.

Observe that the realizations in the Bézout identity can be compactly expressed as

$$\begin{pmatrix} M & Y \\ N & X \end{pmatrix} = \left[\begin{array}{c|cc} A_F & B & -L \\ \hline F & I & 0 \\ C_F & D & I \end{array}\right], \quad \begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} = \left[\begin{array}{c|cc} A_L & -B_L & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]. \tag{9.10}$$

Moreover, we can just read off that

$$\begin{pmatrix} M\tilde{X} & M\tilde{Y} \\ N\tilde{X} & N\tilde{Y}+I \end{pmatrix} = \begin{pmatrix} M & 0 \\ N & I \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \\ 0 & I \end{pmatrix} = \left[\begin{array}{cc|cc} A_F & BF & B & 0 \\ 0 & A_L & -B_L & -L \\ \hline F & F & I & 0 \\ C_F & DF & D & I \end{array}\right]. \tag{9.11}$$

We obtain explicit state-space formulas for all matrices in Theorem 9.13.

Note that these relations were obtained with hardly any computations, in contrast to what you typically see in the literature. Variants (e.g. the extension to time-varying state-space systems) are also easy to obtain.

9.4 Youla Parametrization for Generalized Plants

If $P \in M(F)$ we consider again the generalized plant

$$\begin{pmatrix} z \\ y \end{pmatrix} = P \begin{pmatrix} w \\ u \end{pmatrix}$$

and define the notion of stabilizing controllers as in Lecture 3.

Definition 9.14 *$K \in M(F)$ stabilizes $P \in M(F)$ if $I - P_{22}K$ is invertible and if*

$$\begin{pmatrix} P_{11} & P_{12} & 0 \\ 0 & I & 0 \\ P_{21} & P_{22} & I \end{pmatrix} + \begin{pmatrix} P_{12} \\ I \\ P_{22} \end{pmatrix} K(I - P_{22}K)^{-1} \begin{pmatrix} P_{21} & P_{22} & I \end{pmatrix} \in M(R).$$

To shorten the exposition, we assume that there exists a stabilizing controller for P, and that K stabilizes P iff K stabilizes P_{22}. The Springer-book by Francis (1987) develops algebraic tests for $F = \mathbb{R}(s)$ and $R = RH_\infty$ (as we did in state-space) for these properties.

We can thus parametrize all stabilizing controllers for P, by using the parametrization for $G := P_{22}$ in Theorem 9.13. With the notations from there we get

$$P \star K = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} =$$
$$= P_{11} + P_{12} \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} I & -K \\ -P_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} P_{21} =$$
$$= P_{11} + P_{12}[M\tilde{Y} + MQ\tilde{M}]P_{21}$$

and hence obtain affine dependence of the controlled system on Q.

Our main goal is to obtain this parametrization for LTI generalized plants directly on the basis of state-space descriptions.

Hypothesis 9.15 *Let P be a transfer matrix with*

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad (A, B_2) \text{ stabilizable}, \quad (A, C_2) \text{ detectable.}$$

Choose F, L that render $A + B_2F$ and $A + LC_2$ Hurwitz and define

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & 0 \end{pmatrix} = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + LC_2 & B_1 + LD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right]$$

(in which the lower right transfer matrix indeed vanishes) as well as

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & L & -(B_2 + LD_{22}) \\ \hline -F & 0 & I \\ C_2 + D_{22}F & I & -D_{22} \end{array}\right].$$

Theorem 9.16 *Under Hypotheses 9.15, the LTI controller K stabilizes P iff there exists some $Q \in RH_\infty$ with $\det(I + D_{22}Q(\infty)) \neq 0$ such that*

$$K = J \star Q.$$

The set of all closed-loop transfer matrices $P \star K$ that are achievable by stabilizing controllers is given as

$$\{T_1 + T_2QT_3 : \; Q \in RH_\infty, \quad \det(I + D_{22}Q(\infty)) \neq 0\}.$$

In many (not all!) situations we can assume w.l.o.g. that $D_{22} = 0$ by pushing this direct feedthrough term to the controller. Then the set of stabilized controlled system admits the beautiful affine parametrization

$$T_1 + T_2QT_3 \;\; \text{with free} \;\; Q \in RH_\infty.$$

Remark. The parametrization of all sub-optimal H_∞ in Theorem 7.40 is different: The parameter does not enter the controlled system affinely!

Proof. We can apply Theorem 9.13 with the matrices (9.10) and (9.11) taken for $G = P_{22}$. Note that (9.6) describes stabilizing controllers in $M(\mathbb{R}(s))$, while we target at a parametrization of all stabilizing controllers that are LTI.

If $Q \in RH_\infty$ and $\det(\tilde{X}(\infty) + Q(\infty)\tilde{N}(\infty)) = \det(I + Q(\infty)D_{22}) \neq 0$, then K in (9.6) is obviously proper.

Conversely, if K is LTI and stabilizes P_{22}, we can express it as (9.6) with $Q \in RH_\infty$ such that $\tilde{X} + Q\tilde{N}$ has a rational inverse. Then we get, as in the proof of Theorem 9.13,

$$\begin{pmatrix} \tilde{X} + Q\tilde{N} & \tilde{Y} + Q\tilde{M} \\ 0 & I \end{pmatrix} \begin{pmatrix} M & -K \\ -N & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -N & I \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} \tilde{X} + Q\tilde{N} & \tilde{Y} + Q\tilde{M} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} M & -K \\ -N & I \end{pmatrix} \begin{pmatrix} I & 0 \\ N & I \end{pmatrix}.$$

Hence the inverse of $\tilde{X}+Q\tilde{N}$ is actually proper, which in turn implies $0 \neq \det(\tilde{X}(\infty)+Q(\infty)\tilde{N}(\infty)) = \det(I+Q(\infty)D_{22})$.

The rest of the proof are straightforward computations.

If K is as in (9.6) then $u = Ky$ iff $(\tilde{X}+Q\tilde{N})u = (\tilde{Y}+Q\tilde{M})y$ iff $\tilde{X}u-\tilde{Y}y = Q(-\tilde{N}u+\tilde{M}y)$ iff

$$\begin{pmatrix} \hat{y} \\ \hat{u} \end{pmatrix} = \begin{pmatrix} \tilde{M} & -\tilde{N} \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}, \quad \hat{u} = Q\hat{y}$$

iff (use (9.10))

$$\begin{pmatrix} \hat{y} \\ \hat{u} \end{pmatrix} = \left[\begin{array}{c|cc} A+LC_2 & L & -B-LD_{22} \\ \hline C_2 & I & -D_{22} \\ F & 0 & I \end{array}\right] \begin{pmatrix} y \\ u \end{pmatrix}, \quad \hat{u} = Q\hat{y}$$

iff

$$\begin{pmatrix} \hat{y} \\ u \end{pmatrix} = \left[\begin{array}{c|cc} A+LC_2+BF+LD_{22}F & L & -B-LD_{22} \\ \hline C_2+D_{22}F & I & -D_{22} \\ -F & 0 & I \end{array}\right] \begin{pmatrix} y \\ \hat{u} \end{pmatrix}, \quad \hat{u} = Q\hat{y}.$$

Permuting the outputs shows $u = (J \star Q)y$ and thus $K = J \star Q$.

The interconnection of

$$\begin{pmatrix} \dot{x} \\ \hline z \\ y \end{pmatrix} = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \begin{pmatrix} x \\ \hline w \\ u \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{\xi} \\ u \\ \hat{y} \end{pmatrix} = \left(\begin{array}{c|cc} A+B_2F+LC_2+LD_{22}F & L & -(B_2+LD_{22}) \\ \hline -F & 0 & I \\ C_2+D_{22}F & I & -D_{22} \end{array}\right) \begin{pmatrix} \xi \\ y \\ \hat{u} \end{pmatrix}$$

can be obtained with the feedback $u = \tilde{u}$ from

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \\ \hline z \\ \hat{y} \\ \tilde{u} \end{pmatrix} = \left(\begin{array}{cc|ccc} A & 0 & B_1 & 0 & B_2 \\ LC_2 & A+B_2F+LC_2+LD_{22}F & LD_{21} & -(B_2+LD_{22}) & LD_{22} \\ \hline C_1 & 0 & D_{11} & 0 & D_{12} \\ C_2 & C_2+D_{22}F & D_{21} & -D_{22} & D_{22} \\ 0 & -F & 0 & I & 0 \end{array}\right) \begin{pmatrix} x \\ \xi \\ \hline w \\ \hat{u} \\ u \end{pmatrix}.$$

Without tedious computations, the lower LFT clearly equals

$$\begin{pmatrix} z \\ \hat{y} \end{pmatrix} = \left[\begin{array}{cc|cc} A & -B_2F & B_1 & B_2 \\ LC_2 & A+B_2F+LC_2 & LD_{21} & -B_2 \\ \hline C_1 & -D_{12}F & D_{11} & D_{12} \\ C_2 & C_2 & D_{21} & 0 \end{array}\right] \begin{pmatrix} w \\ \hat{u} \end{pmatrix}$$

which leads with a state-coordinate change to

$$\begin{pmatrix} z \\ \hat{y} \end{pmatrix} = \left[\begin{array}{cc|cc} A+B_2F & -B_2F & B_1 & B_2 \\ 0 & A+LC_2 & B_1+LD_{21} & 0 \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right] \begin{pmatrix} w \\ \hat{u} \end{pmatrix}.$$

With T_1, T_2, T_3 as defined above this is

$$\begin{pmatrix} z \\ \hat{y} \end{pmatrix} = \begin{pmatrix} T_1 & T_2 \\ T_3 & 0 \end{pmatrix} \begin{pmatrix} w \\ \hat{u} \end{pmatrix}.$$

This indeed implies $P \star J \star Q = P \star K = T_1 + T_2QT_3$ as claimed.

■

Remarks

Note that the representations for T_1 and T_2 simplify to

$$T_2 = \left[\begin{array}{c|c} A+B_2F & B_2 \\ \hline C_1+D_{12}F & D_{12} \end{array}\right] \quad \text{and} \quad T_3 = \left[\begin{array}{c|c} A+LC_2 & B_1+LD_{21} \\ \hline C_2 & D_{21} \end{array}\right].$$

Also T_1 can be expressed in the following ways:

$$\left[\begin{array}{cc|c} A+B_2F & -B_2F & B_1 \\ 0 & A+LC_2 & B_1+LD_{21} \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} \end{array}\right] = \left[\begin{array}{cc|c} A+B_2F & -LC_2 & -LD_{21} \\ 0 & A+LC_2 & B_1+LD_{21} \\ \hline C_1+D_{12}F & C_1 & D_{11} \end{array}\right] =$$

$$= T_2 \left[\begin{array}{c|c} A+LC_2 & B_1+LD_{21} \\ \hline -F & 0 \end{array}\right] + \left[\begin{array}{c|c} A+B_2F & B_1 \\ \hline C_1+D_{21}F & D_{11} \end{array}\right] =$$

$$= \left[\begin{array}{c|c} A+B_2F & -L \\ \hline C_1+D_{12}F & 0 \end{array}\right] T_3 + \left[\begin{array}{c|c} A+LC_2 & B_1+LD_{21} \\ \hline C_1 & D_{11} \end{array}\right].$$

Applications

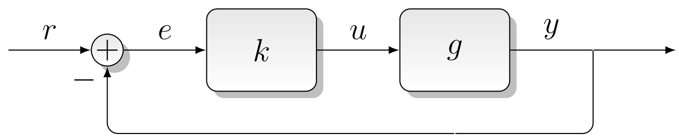
There are very many ways to exploit the Youla parametrization:

- Convex optimization for optimal feedback controller synthesis.
- Switching or interpolating controllers (also on-line for time-varying transitions) with guarantees for stability and performance.
- Designing infinite dimensional controllers for finite-dimensional plants, such as delay system and synthesis of Smith predictors.
- Reduction of control to functional analytic approximation problems for infinite dimensional plants or more general classes of systems.
- Solution of math problems by control techniques (Nehari, Nevanlinna-Pick, Carathéodory-Fejér, ...)

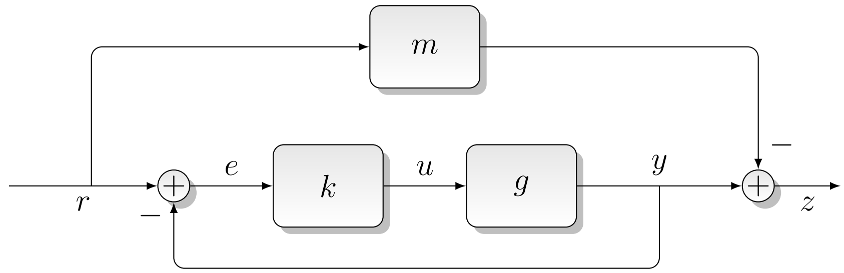
Of course, employing Youla-techniques is often not straightforward.

Exercises

1) Consider the standard tracking configuration given in the figure below with a to-be-controlled stable SISO system g and a to-be-designed controller k



   a) Compute the transfer functions T_{ur} and T_{yr} from r to u and from r to y.
   b) The set V of all strictly proper rational functions without pole on the imaginary axis is a vector space. Argue that the 2-norm of the frequency response of some strictly proper f defined as
   $$\|f\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(i\omega)|^2 d\omega}$$
   comes from an inner product on V.
   c) Show that the subspace of all strictly proper and stable rational functions, denoted as RH_2, is orthogonal to the subspace of all strictly proper and anti stable rational functions, denoted as $RH_2^\perp$ and argue that any strictly proper rational function f without pole on the imaginary axis can be uniquely decomposed into the sum of $f_s \in RH_2$ and $f_u \in RH_2^\perp$ as $f = f_s + f_u$.
   d) Consider now the problem given in the figure below, which is called a model matching problem: Given stable g, m (m strictly proper), design a strictly proper controller k which internally stabilizes the closed loop system and minimizes the 2-norm of the transfer function T_{zr} from r to z, which is called the matching error.



Show that this problem can be reduced to

$$\inf_{q \in RH_2} \|gq - m\|_2. \tag{9.12}$$

e) Give the optimal solution of the reduced and original problem, if g^{-1} is stable and proper, and argue that there exists almost optimal solutions q_ϵ, k_ϵ dependinf on some positive ϵ if g^{-1} is stable but not propber.

f) Suppose that g is stable and has no zeros on the imaginary axis and at infinity. It is a fact that there exists a factorization $g = g_i g_o$, where g_o and g_o^{-1} are proper and stable and g_i has magnitude 1 on the whole imaginary axis (such as e.g. $\frac{s-\alpha_0}{s+\alpha_0}$ for $\alpha_0 > 0$). Use this fact to reduce the problem (9.12) to the equivalent problem

$$\inf_{x \in RH_2} \|g_i x - m\|_2. \tag{9.13}$$

g) Determine the transfer function r such that the optimization problem (9.13) can be transformed to the equivalent problem

$$\inf_{x \in RH_2} \|x - r\|_2.$$

h) Use again the fact thet r can be decomposed into the sum of a strictly proper stable r_s and a strictly proper anti stable r_u to design the optimal controller k_{opt} and to compute $\inf_{x \in RH_2} \|x - r\|_2$.

1) Suppose we have the structured system

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with P_{22} being strictly proper. The aim of this exercise is that the set of all (lower) triangular stabilizing controllers K can be parameterized as

$$K = Y_{11} + Y_{12} Q (I - Y_{22} Q)^{-1} Y_{21} \text{ with } Q = \begin{pmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{pmatrix} \in RH_\infty$$

for some (lower) triangular Y_{11}, Y_{12}, Y_{21} and Y_{22}.

a) Suppose P is stable and let K stabilize P. Show that $Q = K(I - PK)^{-1}$ is a Youla parameter for K. Moreover show that Q is triangular iff K is triangular.

b) Now choose a minimal realization

$$P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & 0 \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

Show that due to $C_1(sI - A)^{-1}B_2 = 0$ the controllable subspace of (A, B_2) is contained in the unobservable subspace of (A, C_1). Show that we can hence find a state coordinate change such that

$$\left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & 0 & 0 \\ A_{21} & A_{22} & B_{22} \\ \hline C_{11} & 0 & 0 \end{array}\right].$$

c) By b) we can realize P as

$$P = \left[\begin{array}{cc|cc} A_{11} & 0 & B_{11} & 0 \\ A_{21} & A_{22} & B_{21} & B_{22} \\ \hline C_{11} & 0 & D_{11} & 0 \\ C_{21} & C_{22} & D_{21} & D_{22} \end{array}\right] \text{ and define } K = \left[\begin{array}{cc|cc} A_{11}^K & 0 & B_{11}^K & 0 \\ A_{21}^K & A_{22}^K & B_{21}^K & B_{22}^K \\ \hline C_{11}^K & 0 & D_{11}^K & 0 \\ C_{21}^K & C_{22}^K & D_{21}^K & D_{22}^K \end{array}\right].$$

Show that K stabilizes P iff (A_{11}, B_{11}), (A_{22}, B_{22}) are stabilizable and (A_{11}, C_{11}), (A_{22}, C_{22}) are detectable.

d) By c) we can find for P with minimal realization as in b) triangular F and L such that $A + BF$ and $A + LC$ are Hurwitz. Define

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \left[\begin{array}{c|cc} A + BF & L & -B \\ \hline -F & 0 & I \\ C & I & 0 \end{array}\right].$$

Show that all stabilizing controllers are given by $K = Y \star Q$ with $Q \in RH_\infty$.

e) Show that if $Q \in RH_\infty$ is triangular then $K = Y \star Q$ is a triangular stabilizing controller for P and show that if K is a triangular stabilizing controller for P then $Q = (I + ZY_{22})^{-1}Z$ with $Z = Y_{12}^{-1}(K - Y_{11})Y_{21}^{-1}$ is the corresponding Youla parameter and is triangular.

10 A Brief Summary and Outlook

We have seen that the **generalized plant framework** is very general and versatile to capture a multitude of interconnection structures. This concept extends to much larger classes of uncertainties, and it can be also generalized to some extent to non-linear systems where it looses, however, its generality.

The **structured singular value** offered us computable tests for robust stability and robust performance against structured linear time-invariant uncertainties. It is possible to include parametric uncertainties which are, however, computationally more delicate.

Nominal controller synthesis, formulated as reducing the H_∞-norm of the performance channel, found a very satisfactory and complete solution in the celebrated Riccati approach to H_∞-control.

Robust controller synthesis is, from a theoretical point of view and despite intensive efforts, still in its infancy. The scaling/controller iteration and variations thereof form - to date - the only possibility to design robustly peforming controllers. It offers no guarantee of optimality, but it has proven useful in practice.

These notes were solely concerned with LTI controller analysis/synthesis, with LTI uncertainties, and with H_∞-norm bounds as performance specifications. **L**inear **M**atrix **I**nequalities (LMIs) techniques offer the possibilities for extensions in the following directions:

- There is a much greater flexibility in choosing the performance specification, such as taking into account H_2-norm constraints (stochastic noise reduction), positive real conditions and amplitude constraints.
- The LMI framework allows the extension to multi-objective design in which the controller is built to guarantee various performance specs on different channels.
- It is possible to include time-varying parametric, non-linear static and non-linear dynamic uncertainties in the robustness tests. The uncertainties will be described in this framework by **I**ntegral **Q**uadratic **C**onstraints (IQCs) which are generalizations of the scalings techniques developed in these notes.
- Finally, LMI techniques allow to perform a systematic design of gain-scheduling controller to attack certain class of non-linear control problems.

A Bisection

At several places we encountered the problem to compute a critical value γ_{critical}, such as in computing the upper bound for the SSV or the optimal value in the H_∞ problem. However, the algorithms we have developed only allowed to test whether a given number γ satisfies $\gamma_{\text{critical}} < \gamma$ or not. How can we compute γ_{critical} just by exploiting the possibility to perform such a test?

The most simple technique is bisection. Fix a level of accuracy $\epsilon > 0$.

- Start with an interval $[a_1, b_1]$ such that $a_1 \leq \gamma_{\text{critical}} \leq b_1$.
- Suppose one has constructed $[a_j, b_j]$ with $a_j \leq \gamma_{\text{critical}} \leq b_j$.

  Then one tests whether
  $$\gamma_{\text{critical}} < \frac{a_j + b_j}{2}.$$
  We assume that this test can be performed and leads to either yes or no as an answer.

  If the answer is **yes**, set $[a_{j+1}, b_{j+1}] = [a_j, \frac{a_j+b_j}{2}]$.

  If the answer is **no**, set $[a_{j+1}, b_{j+1}] = [\frac{a_j+b_j}{2}, b_j]$.
- If $b_{j+1} - a_{j+1} > \epsilon$ then proceed with the second step for j replaced by $j+1$.

  If $b_{j+1} - a_{j+1} \leq \epsilon$ then stop with $a_{j+1} \leq \gamma_{\text{critical}} \leq a_{j+1} + \epsilon$.

Since the length of $[a_{j+1}, b_{j+1}]$ is just half the length of $[a_j, b_j]$, there clearly exists an index for which the length of the interval is smaller than ϵ. Hence the algorithm **always stops**. After the algorithm has stopped, we have calculated γ_{critical} up to the absolute accuracy ϵ.

B Proof of Theorem 7.1

Note that
$$A^T X + XA + XRX + Q = (-A)^T(-X) + (-X)(-A) + (-X)R(-X) + Q$$
and
$$A + RX = -((-A) + R(-X)).$$
Hence we can apply Theorem 7.2 to the ARE/ARI
$$(-A)^T Y + Y(-A) + YRY + Q = 0, \quad (-A)^T Y + Y(-A) + YRY + Q < 0$$
if $(-A, R)$ is stabilizable. This leads to the following result.

Corollary B.1 *Suppose that all hypothesis in Theorem 7.2 hold true but that $(-A, R)$ is only stabilizable. Then the following statements are equivalent:*

(a) H has no eigenvalues on the imaginary axis.

(b) $A^TX + XA + XRX + Q = 0$ has a (unique) anti-stabilizing solution X_+.

(c) $A^TX + XA + XRX + Q < 0$ has a symmetric solution X.

If one and hence all of these conditions are satisfied, then

$$\text{any solution } X \text{ of the ARE or ARI satisfies } \; X \le X_+.$$

Combining this corollary with Theorem 7.2 implies Theorem 7.1.

C Proof of Theorem 7.2

Let us first show the inequality $X_- \le X$ if X_- is the stabilizing solution of the ARE (existence assumed) and X is any other solution of the ARE or ARI.

The key is the easily proved identity

$$\begin{aligned}(A^TZ + ZA + ZRZ + Q) - (A^TY + YA + YRY + Q) \quad &= \\ = (A + RY)^T(Z - Y) + (Z - Y)(A + RY) + (Z - Y)R(Z - Y).& \quad \text{(C.1)}\end{aligned}$$

If we set $Z = X$ and $Y = X_-$, and if we exploit $R \ge 0$, we obtain

$$0 \ge A^TX + XA + XRX + Q \ge (A + RX_-)^T(X - X_-) + (X - X_-)(A + RX_-).$$

Since $A + RX_-$ is stable, we infer that $X - X_- \ge 0$.

Minimality of X_- implies that there is *at most one* stabilizing solution. In fact, if X_1 and X_2 are two stabilizing solutions of the ARE, we can infer $X_1 \le X_2$ (since X_2 is smallest) and $X_2 \le X_1$ (since X_1 is smallest); this leads to $X_1 = X_2$.

Before we can start the proof, we have to establish an important property of the Hamiltonian matrix H. First we observe that the particular structure of H can be expressed as follows. Defining

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

the matrix JH is symmetric! It is a pretty immediate consequence of this fact that the eigenvalues of any Hamiltonian matrix are located symmetrically with respect to the imaginary axis in the complex plane.

Lemma C.1 *Suppose H is a matrix such that JH is symmetric. If H has k eigenvalues in $\mathbb{C}_<$ then it has also k eigenvalues in $\mathbb{C}_>$ (counted with their algebraic multiplicity) and vice versa.*

Proof. By $JH = (JH)^T = H^T J^T = -H^T J$ we infer $JHJ^{-1} = -H^T$. Hence H and $-H^T$ are similar and, thus, their characteristic polynomials are identical: $\det(H - sI) = \det(-H^T - sI)$. Since H has even size, we infer $\det(-H^T - sI) = \det(H^T + sI) = \det(H + sI) = \det(H - (-s)I)$. The resulting equation $\det(H - sI) = \det(H - (-s)I)$ is all what we need: If $\lambda \in \mathbb{C}_<$ is a zero of $\det(H - sI)$ (of multiplicity k), $-\lambda$ is a zero (of multiplicity k) of the same polynomial.

■

Now we can start the stepwise proof of Theorem 7.2.

Proof of (a) $\Rightarrow$ (b). Since H has no eigenvalues in $\mathbb{C}_=$, it has n eigenvalues in $\mathbb{C}_<$ and in $\mathbb{C}_>$ respectively (Lemma C.1). Then there exists a unitary matrix T with

$$T^* H T = \begin{pmatrix} M & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where M of size $n \times n$ is stable and M_{22} of size $n \times n$ is antistable. Let us denote the first n columns of T by Z to infer

$$HZ = ZM.$$

Now partition $Z = \begin{pmatrix} U \\ V \end{pmatrix}$ with two square blocks U and V of size n. The difficult step is now to prove that U is invertible. Then it is not so hard to see that $X_- := VU^{-1}$ is indeed a real Hermitian stabilizing solution of the ARE.

We proceed in several steps. We start by showing

$$V^* U = U^* V.$$

Indeed, $HZ = ZM$ implies $Z^* JHZ = Z^* JZM$. Since the left-hand side is symmetric, so is the right-hand side. This implies $(Z^* JZ)M = M^*(Z^* J^* Z) = -M^*(Z^* JZ)$ by $J^* = -J$. Since M is stable, we infer from $M^*(Z^* JZ) + (Z^* JZ)M = 0$ that $Z^* JZ = 0$[4] what is indeed nothing but $V^* U = U^* V$. Next we show that

$$Ux = 0 \Rightarrow RVx = 0 \Rightarrow UMx = 0.$$

Indeed, $Ux = 0$ and the first row of $HZ = ZM$ imply $UMx = (AU + RV)x = RVx$ and thus $x^* V^* UMx = x^* V^* RVx$. Since $x^* V^* U = x^* U^* V = 0$, the left-hand side and hence

[4]Recall that $AX - XB = 0$ has no nonzero solution iff A and B have no eigenvalues in common.

the right-hand side vanish. By $R \geq 0$, we conclude $RVx = 0$. From $UMx = RVx$ we infer $UMx = 0$. Now we can establish that

$$U \text{ is invertible.}$$

It suffices to prove $\ker(U) = \{0\}$. Let us assume that $\ker(U)$ is nontrivial. We have just shown $x \in \ker(U) \Rightarrow Ux = 0 \Rightarrow UMx = 0 \Rightarrow Mx \in \ker(U)$. Hence $\ker(U)$ is a nonzero M-invariant subspace. Therefore, there exists an eigenvector of M in $\ker(U)$, i.e., an $x \neq 0$ with $Mx = \lambda x$ and $Ux = 0$. Now the second row of $HZ = ZM$ yields $(-QU - A^T V)x = VMx$ and thus $A^T Vx = -\lambda Vx$. Since $Ux = 0$, we have $RVx = 0$ (second step) or $R^T Vx = 0$. Since (A, R) is stabilizable and $\mathrm{Re}(-\lambda) > 0$, we infer $Vx = 0$. Since $Ux = 0$, this implies $Zx = 0$ and hence $x = 0$ because Z has full column rank. However, this contradicts the choice of x as a nonzero vector.

Since U is nonsingular we can certainly define

$$X_- := VU^{-1}.$$

X_- is Hermitian since $V^*U = U^*V$ implies $U^{-*}V^* = VU^{-1}$ and hence $(VU^{-1})^* = VU^{-1}$. X_- is a stabilizing solution of the ARE. Indeed, from $HZ = ZM$ we infer $HZU^{-1} = ZMU^{-1} = ZU^{-1}(UMU^{-1})$. This leads to

$$H \begin{pmatrix} I \\ X_- \end{pmatrix} = \begin{pmatrix} I \\ X_- \end{pmatrix} (UMU^{-1}).$$

The first row of this identity shows $A + RX_- = UMU^{-1}$ such that $A + RX_-$ is stable. The second row reads as $-Q - A^T X_- = X_-(A + RX_-)$ what is nothing but the fact that X_- satisfies the ARE.

So far, X_- might be complex. Since the data matrices are real, we have

$$\overline{A^T X + XA + XRX + Q} = A^T \bar{X} + \bar{X} A + \bar{X} R \bar{X} + Q, \quad \overline{A + RX} = A + R\bar{X}.$$

Consequently, with X_-, also its complex conjugate $\bar{X}_-$ is a stabilizing solution of the ARE. Since we have already shown that there is at most one such solution, we infer $X = \bar{X}_-$ such that X_- must be necessarily real.

Proof of (b) $\Rightarrow$ (a). Just evaluating both sides gives

$$H \begin{pmatrix} I & 0 \\ X_- & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ X_- & I \end{pmatrix} \begin{pmatrix} A + RX_- & R \\ 0 & -(A + RX_-)^T \end{pmatrix}.$$

Hence

$$H \ \text{ is similar to } \begin{pmatrix} A + RX_- & R \\ 0 & -(A + RX_-)^T \end{pmatrix}. \tag{C.2}$$

Therefore, any eigenvalue of H is an eigenvalue of $A + RX_-$ or of $-(A + RX_-)^T$ such that H cannot have eigenvalues in $\mathbb{C}_=$.

Proof of (a) $\Rightarrow$ (c). We perturb Q to $Q + \epsilon I$ and denote the corresponding Hamiltonian as H_ϵ. Since the eigenvalues of a matrix depend continuously on its coefficients, H_ϵ has no eigenvalues in $\mathbb{C}_=$ for all small $\epsilon > 0$. By (b), there exists a symmetric X with $A^T X + XA + XRX + Q + \epsilon I = 0$ what implies (c).

Proof of (c) $\Rightarrow$ (b): Suppose Y satisfies the strict ARI and define $P := A^T Y + YA + YRY + Q < 0$. If we can establish that

$$(A + RY)^T \Delta + \Delta(A + RY) + \Delta R \Delta + P = 0 \tag{C.3}$$

has a stabilizing solution Δ, we are done: Due to (C.1), $X = Y + \Delta$ satisfies the ARE and renders $A + RX = (A + RY) + R\Delta$ stable. The existence of a stabilizing solution of (C.3) is assured as follows: Clearly, $(A + RY, R)$ is stabilizable and (A, P) is observable (due to $P < 0$). Due to $R \geq 0$ and $P < 0$, Lemma 7.3 is applicable and we can conclude that the Hamiltonian matrix corresponding to (C.3) does not have any eigenvalues on the imaginary axis. Hence we can apply the already proved implication (a) $\Rightarrow$ (b) to infer that (C.3) has a stabilizing solution.

■

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